Autonomous Linear Differential Equations with the Hukuhara Derivative that Preserve Polytopes

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By $\Omega(\mathbb{R}^d)$ we denote the set of all nonempty bounded subsets of $\mathbb{R}^d$. For the set of all nonempty compact subset of $\mathbb{R}^d$ we use the notation $K(\mathbb{R}^d)$. By $K_c(\mathbb{R}^d)$ we denote the subset of $K(\mathbb{R}^d)$ that consists of convex sets.

Definition 1. A set $C = A + B \overset{\text{def}}{=} \{a + b : \ a \in A, \ b \in B\}$ is called the *Minkowski sum* of two subsets $A, B \subset \mathbb{R}^d$.

Definition 2 ([1])**.** A set $C \in K_c(\mathbb{R}^d)$ is called the *Hukuhara difference* of $A, B \in K_c(\mathbb{R}^d)$, and denoted by $C = A - B$, if $A = B + C$.

Note that the Hukuhara difference $A - B$ is not defined for any pair of sets $A, B \in K_c(\mathbb{R}^d)$. Moreover, if there exists a set $C \in K_c(\mathbb{R}^d)$ such that $A = B + C$, then, generally speaking, $C \neq A + (-B)$. Indeed, let's take, for example, $A = [0, 2]$ and $B = [0, 1]$. For the set $C = [0, 1]$ we have $A = B + C$. At the same time, $A + (-B) = [-1, 2]$. The following theorem gives a necessary and sufficient condition for the existence of Hukuhara difference between sets $A, B \in K_c(\mathbb{R}^d)$.

Theorem 1 ([2, p. 8])**.** *Let $A, B \in K_c(\mathbb{R}^d)$ be convex compact sets. The Hukuhara difference $A - B$ exists if and only if for any boundary point $a \in \partial A$ there exists at least one point $c \in \mathbb{R}^d$ such that*

$$a \in B + \{c\} \subset A.$$

If the Hukuhara difference $A - B$ exists, then it is unique. This statement can be derived from the following lemma.

Lemma 1 ([2, p. 10])**.** *Let $C \subset \mathbb{R}^d$, $D \in K_c(\mathbb{R}^d)$, $B \in \Omega(\mathbb{R}^d)$, and $C + B \subset D + B$. Then $C \subset D$.*

By $\overline{\mathbb{B}}^d \overset{\text{def}}{=} \{x \in \mathbb{R}^d : \ \|x\| \leq 1\}$ we denote the closed ball of unit radius centered at the origin.

Definition 3. *Hausdorff distance* $h(A, B)$ between sets A and $B \in \Omega(\mathbb{R}^d)$ is defined as

$$h(A, B) \overset{\text{def}}{=} \inf_{r \geq 0} \big\{r : \ A \subset B + r\overline{\mathbb{B}}^d, \ B \subset A + r\overline{\mathbb{B}}^d\big\}.$$

It follows directly from the definition that $h(A, B) = 0$ iff $\overline{A} = \overline{B}$. According to Hahn's theorem the pair $(K(\mathbb{R}^d), h)$ is a complete separable metric space, and $K_c(\mathbb{R}^d)$ is its closed subset.

By $I \subset \mathbb{R}$ we denote an arbitrary open interval that may be unbounded.

Definition 4. A map $X \colon I \to K_c(\mathbb{R}^d)$ is called *differentiable by Hukuhara* at $t_0 \in I$ if the limits

$$\lim_{\Delta t \to +0} \frac{X(t_0 + \Delta t) - X(t_0)}{\Delta t} \quad \text{and} \quad \lim_{\Delta t \to +0} \frac{X(t_0) - X(t_0 - \Delta t)}{\Delta t}$$

both exist and are equal to the same convex compact set $D_H X(t_0)$, that is called *Hukuhara derivative* of X at t_0.

It is easy to see that if the map $X : I \to K_c(\mathbb{R}^d)$ is differentiable at every point of I, then for any $a < b \in I$ the difference $X(b) - X(a)$ is defined. Therefore, according to Theorem 1, non-decreasing of $\operatorname{diam} X(\,\cdot\,)$ is a necessary condition for the existence of Hukuhara derivative $D_H X(t)$, $t \in I$.

For a given positive integer $d \in \mathbb{N}$, let us consider the linear differential equation

$$D_H X(t) = A(t)X(t), \quad X(t) \in K_c(\mathbb{R}^d), \quad t \geq 0, \tag{1}$$

with a semi-continuous coefficient $d \times d$ matrix. By a polytope we mean a convex hull of finite number of points in $\mathbb{R}^d$.

Definition 5. We say that equation (1) *preserves polytopes* if for any its solution $X(\,\cdot\,)$ such that $X(0)$ is a polytope, it follows that $X(t)$ is a polytope for all $t \geq 0$.

Let us consider the problem of obtaining a necessary and sufficient condition for equation (1) to preserve polytopes. This problem is partially solved, namely, we obtained the complete description of the autonomous differential equations (1) that posses this property.

Theorem 2. *Equation* (1) *with a constant coefficient matrix $A(\,\cdot\,) \equiv A$ preserves polytopes if and only if there exists a real number λ and non-negative integers $a < b$ such that $A^b = \lambda A^a$.*

References

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[2] V. Lakshmikantham, T. G. Bhaskar, J. Vasundhara Devi, *Theory of Set Differential Equations in Metric Spaces*. Cambridge Scientific Publishers, Cambridge, 2006.