Asymptotic Behavior of Stochastic Functional Differential Equations in Hilbert Spaces

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We study the long time behavior of nonlinear stochastic functional-differential equations in Hilbert spaces. In particular, we start with establishing the existence and uniqueness of mild solutions. We proceed with deriving a priory uniform in time bounds for the solutions in the appropriate Hilbert spaces. These bounds enable us to establish the existence of invariant measure based on the Krylov–Bogoliubov theorem on the tightness of the family of measures.

1 Introduction

In this work we study the asymptotic behaviour of solutions of stochastic functional-differential equations. In a bounded domain, the equation reads as

$$\begin{gathered} du=[Au+f(u_t)]\,dt+\sigma(u_t)\,dW(t)\ \text{ in } D,\ \ t>0; \\ u(t,x)=\phi(t,x),\ \ t\in[-h,0),\quad u(0,x)=\varphi_0(x)\ \text{ in } D; \\ u(t,x)=0,\ \ x\in\partial D,\ \ t\geq 0. \end{gathered} \tag{1.1}$$

The corresponding problem in the entire space has the form

$$\begin{gathered} du=[Au+f(u_t)]\,dt+\sigma(u_t)\,dW(t)\ \text{ in } \mathbb{R}^d,\ \ t>0; \\ u(t,x)=\phi(t,x),\ \ t\in[-h,0),\quad u(0,x)=\varphi_0(x)\ \text{ in } \mathbb{R}^d. \end{gathered} \tag{1.2}$$

Here A is an elliptic operator

$$A=A(x)=\sum_{i,j=1}^d a_{ij}(x)\,\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i(x)\,\frac{\partial}{\partial x_i}+c(x), \tag{1.3}$$

the interval $[-h,0]$ is the interval of delay, and $u_t=u(t+\theta)$ with $\theta\in[-h,0)$.

Functional differential equations of types (1.1) and (1.2) are mathematical models of processes, the evolution of which depends on the previous states. The classical results for deterministic functional-differential equations in finite dimensional spaces can be found in [6] and the references therein. Stochastic functional differential equations in finite dimensions have been studied extensively as well. In particular, the existence of invariant measures for stochastic ordinary differential equations was established in [1] and [5].

The results on functional differential equations in infinite dimensions are significantly more sparse.

The main goal of the present work is to establish the existence and uniqueness of invariant measures for equations (1.1) and (1.2) based on the Krylov–Bogoliubov theorem on the tightness of the family of measures [7]. More precisely, we will use the compactness approach of Da Parto and Zabczyk [3], which involves the following key steps:

(i) Establishing the existence of a Markovian solution of (1.1) or (1.2) in a certain functional space, in which the corresponding transition semigroup is Feller;

(ii) Showing that the semigroup $S(t)$ generated by A is compact;

(iii) Showing that the corresponding equation with a suitable initial condition has a solution, which is bounded in probability.

This approach was used in establishing the existence of invariant measure for a large class of stochastic nonlinear partial differential equations without delay, e.g. [4] and references therein. For functional differential equations in finite dimensions, the approach above was used in [2]. In this work, the author established the existence of an invariant measure in $\mathbb{R}^d \times L^2(-h,0;\mathbb{R}^d)$. In contrast, for stochastic partial differential equations, the natural phase space for the mild solutions of (1.2) is $L^2_\rho(\mathbb{R}^d) \times L^2(-h,0,L^2_\rho(\mathbb{R}^d))$, where $L^2_\rho(\mathbb{R}^d)$ is a weighted space. The equations of type (1.1) and (1.2) were studied in the space $C(-1,0,L^2_\rho(\mathbb{R}^d))$, which is a significantly easier problem. In these spaces the authors studied the conditions for the existence and uniqueness of a solution, as well as their Markov's and Feller properties. However, in order to apply the compactness approach one needs to work in $L^2_\rho(\mathbb{R}^d) \times L^2(-h,0,L^2_\rho(\mathbb{R}^d))$, which is done in this work.

2 Formulation of the problem and the main result

Throughout the paper, the domain D is either a bounded domain with ∂D satisfying the Lyapunov condition, or $D=\mathbb{R}^d$. Denote

$$\rho(x) := \frac{1}{1+|x|^r}\,,$$

where $r>d$ if $D=\mathbb{R}^d$ and $r=0$ (i.e. no weight) for bounded D. We introduce the following spaces:

$$B_0^\rho := L^2_\rho(D), \quad B_1^\rho := L^2(-h,0,L^2_\rho(D)), \quad B^\rho := B_0^\rho \times B_1^\rho, \quad H := L^2(D).$$

The coefficients a_{ij} of the operator A defined in (1.3) are Holder continuous with the exponent $\beta \in (0,1)$, symmetric, bounded and satisfying the elipticity condition

$$\sum_{i,j=1}^d a_{i,j}\eta_i\eta_j \geq C_0|\eta|, \quad \eta \in \mathbb{R}^d$$

for some $C_0>0$. The coefficients b_i and c are also bounded and Holder continuous with some positive Holder exponent.

If D is bounded, we impose homogeneous Dirichlet boundary conditions on ∂D. In this case,

$$D(A) = H^2(D) \cap H^1_0(D).$$

If $D=\mathbb{R}^d$, then $D(A)=H^2(\mathbb{R}^d)$. Denote $G(t,x,y)$ to be the fundamental solution (or the Green's function in the case of bounded D) for $\frac{\partial}{\partial t} - A$. It is well known, that there are positive constants $C_1(T), C_2(T) > 0$ such that

$$0 \leq G(t,x,y) \leq C_1(T) t^{-d/2} e^{-C_2(T)\,\frac{|x-y|^2}{t}} \tag{2.1}$$

for $t \in [0,T]$ and $x,y \in D$. Note that in (2.1), C_1 and C_2 depend not only on T, but on the constants C_0, d, T, maximum values of the coefficients of A, and the Holder constants. If the operator is in the divergence form $Au = \operatorname{div}(a\nabla u)$, the estimates are of a different type, namely,

$$g_1(t,x-y) \leq G(t,x,y) \leq g_2(t,x-y),$$

where

$$g_i(t,x)=K(C_0,d)t^{-d/2}e^{-K(C_0,d)\,\frac{|x|^2}{t}},\ \ i=1,2,\ \ t\geq 0,\ \ x,y\in\mathbb{R}^d.$$

In this case, in contrast with (2.1), the constant $K(C_0,d)$ is independent of t.

Let $\sum\limits_{i=1}^{\infty} a_i<\infty$, and e_n be orthonormal basis in H, such that $e_n\in L^\infty(D)$ and $\sup\limits_n \|e_n\|_{L^\infty(D)}<\infty$. Introduce the operator $Q\in\mathcal{L}(H)$ such that Q is non-negative, $\mathrm{Tr}(Q)<\infty$, $Qe_n=a_ne_n$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space. Introduce

$$W(t):=\sum_{i=1}^{\infty}\sqrt{a_i}\beta_i(t)e_i(x),\ \ t\geq 0,$$

which is a Q-Wiener process on $t\geq 0$ with values in $L^2(Q)$. Here $\beta_i(t)$ are standard, one dimensional, independent Wiener processes. Also let $\{F_t,\ t\geq 0\}$ be a normal filtration satisfying

- $W(t)$ is $\mathcal{F}_t$-measurable;

- $W(t+h)-W(t)$ is independent of $\mathcal{F}_t$ $\forall\,h\geq 0,t\geq 0$.

Denote $U=Q^{\frac{1}{2}}(H)$. It is well known $U\in L^\infty(D)$. Introduce the multiplication operator $\Phi:U\to B_0^\rho$ as follows: for fixed $\varphi\in B_0^\rho$, let $\Phi(\psi):=\varphi\psi$, $\psi\in U$. Since $\varphi\in B_0^\rho$ and $\varphi\in L^\infty(D)$, the operator is well defined and hence $\Phi\circ Q^{1/2}:L^2(D)\to B_0^\rho$ defines a Hilbert–Schmidt operator. The operator Φ is also a Hilbert–Schmidt operator satisfying

$$\|\Phi\circ Q^{1/2}\|^2_{\mathcal{L}_2}=\sum_{n=1}^{\infty}\|\Phi\circ Q^{1/2}e_n\|^2_{B_0^\rho}=\sum_{n=1}^{\infty}a_n\int\limits_D\varphi^2(x)e_n^2(x)\rho(x)\,dx\leq \mathrm{Tr}(Q)\sup_n\|e_n\|^2_\infty\|\varphi\|^2_\rho,$$

where $\mathrm{Tr}(Q)=\sum_{n=1}^\infty a_n=a$. Hence if $\Phi:\Omega\times[0,T]\to\mathcal{L}(U,B_0^\rho)$ is a predictable process satisfying

$$\mathbb{E}\int\limits_0^T\|\Phi\circ Q^{1/2}\|^2_{\mathcal{L}_2}\,ds<\infty,$$

following [3] we can define

$$\int\limits_0^t\Psi(s)\,dW(s)\in B_0^\rho$$

with the following expansion

$$\int\limits_0^t\Psi(s)\,dW(s)=\sum_{i=1}^{\infty}\sqrt{a_i}\int\limits_0^t\Phi(s,\,\cdot\,)e_i(\,\cdot\,)\,d\beta_i(s).$$

Furthermore,

$$\mathbb{E}\bigg\|\int\limits_0^t\Psi(s)\,dW(s)\bigg\|^2_\rho\leq a\sup_n\|e_n\|^2_\infty\int\limits_0^t\mathbb{E}\|\Psi(s,\,\cdot\,)\|^2_{B_0^\rho}\,ds.$$

Assumptions on nonlinearities

Assume f and σ satisfy the following conditions:

(i) The functionals f and σ map B_1^ρ to B_0^ρ;

(ii) There exists a constant $L > 0$ such that
$$\|f(\varphi_1) - f(\varphi_2)\|_{B_0^\rho} + \|\sigma(\varphi_1) - \sigma(\varphi_2)\|_{B_0^\rho} \le L\|\varphi_1 - \varphi_2\|_{B_1^\rho}$$
for any $\varphi_1, \varphi_2 \in B_1^\rho$.

Definition. An $\mathcal{F}_t$ measurable random process $u(t,\,\cdot\,) \in B_0^\rho$ is a mild solution of (1.1) or (1.2) if
$$u(t,\,\cdot\,) = S(t)\varphi(0,\,\cdot\,) + \int\limits_0^t S(t-s)f(u_s)\,ds + \int\limits_0^t S(t-s)\sigma(u_s)\,dW(s), \tag{2.2}$$
where
$$u(0,\,\cdot\,) = \varphi(0,\,\cdot\,) \in B_0^\rho, \;\; u(t,\,\cdot\,) = \varphi(t,\,\cdot\,) \in B_1^\rho, \;\; t \in [-h, 0].$$

Theorem 1 (Existence and uniqueness)**.** *Suppose f and σ satisfy conditions* (i) *and* (ii)*, and $\varphi(t,\,\cdot\,)$ is an $\mathcal{F}_0$ measurable random process for $t \in [-h, 0]$, which is independent of W and such that*
$$\mathbb{E}\|\varphi(0,\,\cdot\,)\|_{B_0^\rho}^p < \infty \;\; \text{and} \;\; \mathbb{E}\|\varphi(\,\cdot\,,\,\cdot\,)\|_{B_1^\rho}^p < \infty, \;\; p \ge 2.$$
Then there exists a unique mild solution of (1.1) (*or* (1.2)) *on* $[0,T]$*, and*
$$\mathbb{E}\|y(t)\|_{B^\rho}^p \le K(T)\big(1 + \mathbb{E}\|y(0)\|_{B^\rho}^p\big), \;\; t \in [0,T].$$

Define $\overline{\rho}(x) = (1 + |x|^{\overline{r}})^{-1}$. The main result of the paper is the following theorem.

Theorem 2. *Let the assumptions of Theorem 1 hold. Assume equation* (2.2) *has a solution in $B^{\overline{\rho}}$ which is bounded in probability for $t \ge 0$ with $r > d + \overline{r}$. Then there exists an invariant measure μ on B^ρ, i.e.*
$$\int\limits_{B^\rho} P_t\varphi(x)\,d\mu(x) = \int\limits_{B^\rho} \varphi(x)\,d\mu \;\; \text{for any} \;\; t \ge 0 \;\; \text{and} \;\; \varphi \in C_b(B^\rho).$$

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