Application of the Averaging Method to Solving Boundary Value Problems for Systems with Impulse Action at Non-Fixed Moments of Time

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The averaging method is applied to study the existence of solutions of boundary value problems for systems with impulse action at non-fixed moments of time. It is shown that if an averaged boundary value problem has a solution, then the original problem is solvable as well. Here the averaged system is a system of autonomous ordinary differential equations.

1 Introduction

The present paper deals with the following boundary value problem for a system of differential equations with impulse action at non-fixed moments of time:

$$\begin{gathered} \dot{x} = \varepsilon X(t,x), \ \ t \neq t_i(x), \\ \Delta x\big|_{t=t_i(x)} = \varepsilon I_i(x), \\ F\Big(x(0), x\Big(\frac{T}{\varepsilon}\Big)\Big) = 0. \end{gathered} \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter, $t_i(x) < t_{i+1}(x)$, $i = 1, 2, \dots$, are the moments of impulse, X and I_i are d-dimensional vector functions.

Assuming that there exist the limits

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T X(t,x)\,dt \tag{1.2}$$

and

$$I_0(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{0<t_i(x)<T} I_i(x), \tag{1.3}$$

we put problem (1.1) in correspondence with the averaged boundary value problem

$$\dot{y} = \varepsilon\big[X_0(y) + I_0(y)\big], \quad F\Big(y(0), y\Big(\frac{T}{\varepsilon}\Big)\Big) = 0, \tag{1.4}$$

or, on the slow time scale $\tau = \varepsilon t$,

$$\frac{dy}{d\tau} = X_0(y), \quad F(y(0), y(T)) = 0.$$

The main result of this paper is a proof of the following statement: if the averaged boundary value problem has a solution, then, for small values of parameter ε, the original boundary value problem (1.1) also has a solution, and there is a proximity between their solutions.

Boundary value problems for systems with impulse action have been considered by many authors. To our knowledge, these problems were first studied in [3] when investigating periodical solutions by using the Samoilenko numerical-analytic method. Boundary value problems for systems with non-fixed moments of impulse were studied in [1] for the case of linear boundary conditions, and in [2] for the nonlinear case.

In the theory of ordinary differential equations, the method of averaging was first applied to boundary value problems in [4]. This method made it possible to reduce a boundary value problem for a non-autonomous system to an analogous problem for an autonomous averaged system. In the present paper, we apply this idea to solving the boundary value problem (1.1).

2 Formulation of the problem and the main result

We consider problem (1.1) under the assumption that the following conditions are satisfied:

(1) The functions $X(t,x)$ and $I_i(x)$ are uniformly continuous in a domain $Q = \{t \geq 0,\ x \in D \subset \mathbb{R}^d\}$;

(2) The functions $X(t,x)$ and $I_i(x)$ are bounded by a constant $M > 0$ and, with respect to x, satisfy the Lipschitz condition with a constant $L > 0$;

(3) There exist uniform in $x \in D$ limits (1.2) and (1.3), as well as the limits

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \frac{\partial X(t,x)}{\partial x}\, dt = \frac{\partial X_0(x)}{\partial x}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{0<t_i<T} \frac{\partial I_i(x)}{\partial x} = \frac{\partial I_0(x)}{\partial x}\,;$$

(4) There exists a constant $C > 0$ such that, for $t \geq 0$ and $x \in D$,

$$i(t,x) \leq Ct,$$

where $i(t,x)$ is the number of impulses on $(0,t)$, and

$$\inf_{x\in D} \tau_{k+1}(x) > \sup_{x\in D} \tau_k(x);$$

(5) The averaged problem (1.4) has a solution $y = y(\tau) = y(\varepsilon,\tau)$ that belongs to D together with some ρ-neighborhood, in which $F(x,y)$ has uniformly continuous partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, and $\det \frac{\partial F_0(x_0)}{\partial x_0} \neq 0$, here $x_0 = y(0)$, $F_0(x_0) = F(x_0, y(T,x_0))$.

Theorem 1. *Let conditions* (1)–(5) *be satisfied. Then there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$ *one can specify a function* $\xi = \xi(\varepsilon)$, $\varepsilon \to 0$, *such that the boundary value problem* (1.1) *has a unique solution* $x(t, \varepsilon)$ *in* $\xi(\varepsilon)$*-neighborhood of* $y(\varepsilon t)$, *i.e.,*

$$\big|x(t,\varepsilon) - y(\varepsilon t)\big| < \xi(\varepsilon), \quad t \in \Big[0, \frac{T}{\varepsilon}\Big], \quad \varepsilon \in (0, \varepsilon_0).$$

The outline of the proof is as follows.

I. We first consider the system with impulse effect at fixed moments t_i on $[0, \frac{T}{\varepsilon}]$:

$$\begin{aligned} &\dot{x} = \varepsilon X(t,x), \quad t \neq t_i, \\ &\Delta x\big|_{t=t_i} = \varepsilon I_i(x(t_i)). \end{aligned} \tag{2.1}$$

For this system, we derive a variational equation linearized along its solution $x(t, x_0)$ ($x(0, x_0) = 0$), i.e.,

$$\begin{aligned} \dot{z} &= \varepsilon\,\frac{\partial X(t, x(t,x_0))}{\partial x}\,, \quad t \neq t_i, \\ \Delta z\big|_{t=t_i} &= \varepsilon\,\frac{\partial I_i(x(t_i, x_0))}{\partial x}\,z(t_i), \end{aligned} \tag{2.2}$$

where $z(t) = \frac{\partial X(t,x_0)}{\partial x_0}$. We then establish the proximity between the solution of (2.2) and the solution $\frac{\partial y(\varepsilon t, x_0)}{\partial x_0}$ of the variational equation for the averaged system (under respective initial conditions).

II. By using the implicit function theorem, we prove the existence and uniqueness of a solution of the boundary value problem for system (2.1).

III. Let us fix p points $y^1, y^2, \dots, y^p$ in some neighborhood of a solution of the averaged problem and consider the following boundary value problem:

$$\begin{aligned} &\dot{x} = \varepsilon X(t,x), \quad t \neq t_i(y^i), \\ &\Delta x\big|_{t=t_i(y^i)} = \varepsilon I_i(y^i), \\ &F\Big(x(0), x\Big(\frac{T}{\varepsilon}\Big)\Big) = 0. \end{aligned}$$

From what has been proved above, we conclude that this boundary value problem, for ε small enough, has a unique solution $x(t, y^1, \dots, y^p)$. If we choose $y^1, \dots, y^p$ so that

$$y^i = x(t_i(y^i), y^1, \dots, y^p), \quad i = \overline{1,p}, \tag{2.3}$$

then the function $x(t, y^1, \dots, y^p)$ is the desired solution of problem (1.1). Using a fixed-point theorem, we show that system (2.3) has a solution. This completes the proof.

References

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