Boundary Value Problems for Multi-Term Fractional Differential Equations with ϕ-Laplacian at Resonance

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1 Introduction

Let $T>0$ be given, $J=[0,T]$, $X=C(J)\times\mathbb{R}$ and $\|x\|=\max\{|x(t)|:\ t\in J\}$ be the norm in $C(J)$.

Let ϕ be an increasing and odd homeomorphism with $\phi(\mathbb{R})=\mathbb{R}$. The special case of ϕ is p-Laplacian $\phi_p(x)=|x|^{p-2}x$, $p>1$.

We discuss the fractional boundary value problem

$$
{}^c\!D^{\alpha}\phi\big({}^c\!D^{\beta}x(t)-a(t){}^c\!D^{\gamma_1}x(t)-b(t){}^c\!D^{\gamma_2}x(t)\big)=f(t,x(t)), \tag{1.1}
$$

$$
x(0)=x(T),\quad {}^c\!D^{\beta}x(t)\big|_{t=0}=0, \tag{1.2}
$$

where $\alpha\in(0,1]$, $0<\gamma_2<\gamma_1<\beta\le 1$, $a,b\in C(J)$, $f\in C(J\times\mathbb{R})$ and ${}^c\!D$ denotes the Caputo fractional derivative.

Definition 1.1. We say that $x:J\to\mathbb{R}$ is *a solution of equation* (1.1) if $x,{}^c\!D^{\beta}x\in C(J)$ and (1.1) holds for $t\in J$. A solution x of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

We recall the definitions of the Riemann-Liouville fractional integral and the Caputo fractional derivative [2, 3].

The Riemann–Liouville fractional integral $I^{\gamma}x$ of order $\gamma>0$ of a function $x:J\to\mathbb{R}$ is defined as

$$
I^{\gamma}x(t)=\int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\,x(s)\,\mathrm{d}s,
$$

where Γ is the Euler gamma function. I^0 is the identical operator.

The Caputo fractional derivative ${}^c\!D^{\gamma}x$ of order $\gamma>0$, $\gamma\notin\mathbb{N}$, of a function $x:J\to\mathbb{R}$ is given as

$$
{}^c\!D^{\gamma}x(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t\frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)}\Big(x(s)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}\,s^k\Big)\,\mathrm{d}s,
$$

where $n=[\gamma]+1$, $[\gamma]$ means the integral part of the fractional number γ. If $\gamma\in\mathbb{N}$, then ${}^c\!D^{\gamma}x(t)=x^{(\gamma)}(t)$. In particular,

$$
{}^c\!D^{\gamma}x(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(0))\,\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}\,I^{1-\gamma}(x(t)-x(0)),\ \ \gamma\in(0,1).
$$

It is well known that $I^\gamma : C(J) \to C(J)$ for $\gamma \in (0,1)$; $I^\gamma I^\mu x(t) = I^{\gamma+\mu} x(t)$ for $x \in C(J)$ and $\gamma, \mu \in (0,\infty)$; ${}^c\!D^\gamma I^\gamma x(t) = x(t)$ for $x \in C(J)$ and $\gamma > 0$; if $x, {}^c\!D^\gamma x \in C(J)$ and $\gamma \in (0,1)$, then $I^\gamma {}^c\!D^\gamma x(t) = x(t) - x(0)$; if $0 < \beta < \alpha < 1$ and $x, {}^c\!D^\alpha x \in C(J)$, then ${}^c\!D^\beta x = I^{\alpha-\beta} {}^c\!D^\alpha x$.

Problem (1.1), (1.2) is at resonance, because every constant function x on J is a solution of problem ${}^c\!D^\alpha \phi({}^c\!D^\beta x - a(t) {}^c\!D^{\gamma_1} x - b(t) {}^c\!D^{\gamma_2} x) = 0$, (1.2).

The aim of this paper is to study the existence of solutions to problem (1.1), (1.2). To this end we first introduce an operator $\mathcal{Q} : C(J) \to C(J)$. Then, by $\mathcal{Q}$ an operator $\mathcal{L} : X \to X$ is defined and it is proved that if $(x,c) \in X$ is a fixed point of $\mathcal{L}$, then x is a solution of problem (1.1), (1.2). The existence of a fixed point of $\mathcal{L}$ is proved by the Schaefer fixed point theorem [1, 4].

We work with the following conditions for a, b and f in (1.1):

(H_1) $a(t) \geq 0$, $b(t) \geq 0$ for $t \in J$.

(H_2) There exist $D, H \in \mathbb{R}$, $D < 0 < H$, such that

$$\begin{aligned} f(t,x) < 0 \ \text{ for } \ t \in J, \ \ x \leq D, \\ f(t,x) > 0 \ \text{ for } \ t \in J, \ \ x \geq H. \end{aligned}$$

(H_3) There exists a nondecreasing function $w : [0,\infty) \to (0,\infty)$ such that

$$\lim_{v \to \infty} \frac{1}{v}\, \phi^{-1}\Big(\frac{T^\alpha w(v)}{\Gamma(\alpha+1)}\Big) = 0$$

and

$$|f(t,x)| \leq w(|x|) \ \text{ for } \ (t,x) \in J \times \mathbb{R},$$

where ϕ^{-1} is the inverse function of ϕ.

2 Operator $\mathcal{Q}$ and its properties

The following result is the generalization of the Gronwall–Bellman lemma for singular kernels.

Lemma 2.1. *Let $0 < \zeta < \rho \leq 1$, $z \in C(J)$ be nonnegative and $c_1, c_2 \in [0,\infty)$. Suppose that $v \in C(J)$ is nonnegative and*

$$v(t) \leq z(t) + c_1 I^\zeta v(t) + c_2 I^\rho v(t), \ \ t \in J.$$

Then

$$v(t) \leq z(t) + d\Big(c_1 + \frac{c_2 \Gamma(\zeta) T^{\rho-\zeta}}{\Gamma(\rho)}\Big) I^\zeta z(t), \ \ t \in J,$$

where $d = d(\zeta,\rho)$ is a positive constant.

Let $\mathcal{F} : C(J) \to C(J)$ be the Nemytskii operator associated to f,

$$\mathcal{F}x(t) = f(t, x(t)).$$

For $x \in C(J)$, we discuss the auxiliary equation

$$u(t) = a(t) I^{\beta-\gamma_1} u(t) + b(t) I^{\beta-\gamma_2} u(t) + \phi^{-1} I^\alpha \mathcal{F}x(t) \tag{2.1}$$

with the unknown function u.

The following result is established by using Lemma 2.1 and the Schaefer fixed point theorem in $C(J)$.

Lemma 2.2. *Let $x \in C(J)$. Then equation* (2.1) *has a unique solution u in the set $C(J)$.*

Keeping in mind Lemma 2.2, for every $x \in C(J)$ there exists a unique solution $u \in C(J)$ of equation (2.1). We put $\mathcal{Q}x = u$ and have an operator $\mathcal{Q} : C(J) \to C(J)$ satisfying

$$\mathcal{Q}x(t) = a(t)I^{\beta-\gamma_1}\mathcal{Q}x(t) + b(t)I^{\beta-\gamma_2}\mathcal{Q}x(t) + \phi^{-1}I^{\alpha}\mathcal{F}x(t), \ \ x \in C(J). \tag{2.2}$$

The properties of $\mathcal{Q}$ are given in the following two lemmas.

Lemma 2.3. *Let* (H_1) *and* (H_2) *hold. Then*

$$\begin{aligned} x \in C(J), \ \ x(t) \le D \ \ \text{on } J &\implies \mathcal{Q}x(t) < 0 \ \ \text{on } (0,T],\\ x \in C(J), \ \ x(t) \ge H \ \ \text{on } J &\implies \mathcal{Q}x(t) > 0 \ \ \text{on } (0,T], \end{aligned}$$

Lemma 2.4. *Let* (H_3) *hold. Then $\mathcal{Q} : C(J) \to C(J)$ is continuous and*

$$\|\mathcal{Q}x\| \le E\phi^{-1}\Big(\frac{T^{\alpha}w(\|x\|)}{\Gamma(\alpha+1)}\Big), \ \ x \in C(J), \tag{2.3}$$

where

$$E = 1 + \frac{T^{\beta-\gamma_1}}{\Gamma(\beta-\gamma_1+1)}\Big(\|b\| + \frac{\|a\|\Gamma(\beta-\gamma_1)T^{\gamma_1-\gamma_2}}{\Gamma(\beta-\gamma_2)}\Big).$$

3 Operator $\mathcal{L}$ and its properties

Let an operator $\mathcal{L} : X \to X$ be defined by

$$\mathcal{L}(x,c) = \Big(c + I^{\beta}\mathcal{Q}x(t), c - I^{\beta}\mathcal{Q}x(t)\big|_{t=T}\Big).$$

The following two lemmas give the properties of $\mathcal{L}$.

Lemma 3.1. *If (x,c) is a fixed point of $\mathcal{L}$, then x is a solution to problem* (1.1), (1.2).

Proof. Let $(x,c) = \mathcal{L}(x,c)$ for some $(x,c) \in X$. Then

$$\begin{gathered} x(t) = c + I^{\beta}\mathcal{Q}x(t), \ \ t \in J,\\ I^{\beta}\mathcal{Q}x(t)\big|_{t=T} = 0, \end{gathered}$$

and therefore $x(0) = c$, $x(T) = c$ and ${}^{c}\!D^{\beta}x(t) = \mathcal{Q}x(t)$ for $t \in J$. Hence ${}^{c}\!D^{\beta}x \in C(J)$ and since $\mathcal{Q}x(t)|_{t=0} = 0$, we have ${}^{c}\!D^{\beta}x(t)|_{t=0} = 0$. Thus x satisfies the boundary condition (1.2) and

$${}^{c}\!D^{\gamma_1}x(t) = I^{\beta-\gamma_1}\,{}^{c}\!D^{\beta}x(t), \quad {}^{c}\!D^{\gamma_2}x(t) = I^{\beta-\gamma_2}\,{}^{c}\!D^{\beta}x(t), \ \ t \in J.$$

Combining these equalities with (2.2) and ${}^{c}\!D^{\beta}x(t) = \mathcal{Q}x(t)$ we obtain

$$\begin{aligned} {}^{c}\!D^{\beta}x(t) = \mathcal{Q}x(t) &= a(t)I^{\beta-\gamma_1}\mathcal{Q}x(t) + b(t)I^{\beta-\gamma_2}\mathcal{Q}x(t) + \phi^{-1}I^{\alpha}\mathcal{F}x(t)\\ &= a(t)I^{\beta-\gamma_1}\,{}^{c}\!D^{\beta}x(t) + b(t)I^{\beta-\gamma_2}\,{}^{c}\!D^{\beta}x(t) + \phi^{-1}I^{\alpha}\mathcal{F}x(t)\\ &= a(t)\,{}^{c}\!D^{\gamma_1}x(t) + b(t)\,{}^{c}\!D^{\gamma_2}x(t) + \phi^{-1}I^{\alpha}\mathcal{F}x(t), \ \ t \in J. \end{aligned}$$

In particular,

$${}^{c}\!D^{\beta}x(t) - a(t)\,{}^{c}\!D^{\gamma_1}x(t) - b(t)\,{}^{c}\!D^{\gamma_2}x(t) = \phi^{-1}I^{\alpha}\mathcal{F}x(t), \ \ t \in J.$$

Applying ϕ and then ${}^{c}\!D^{\alpha}$ on both its sides, it follows

$${}^{c}\!D^{\alpha}\phi\Big({}^{c}\!D^{\beta}x(t) - a(t)\,{}^{c}\!D^{\gamma_1}x(t) - b(t)\,{}^{c}\!D^{\gamma_2}x(t)\Big) = \mathcal{F}x(t), \ \ t \in J.$$

Hence x is a solution of equation (1.1). As a result, x is a solution to problem (1.1), (1.2). □

Lemma 3.2. *Let* (H_3) *hold. Then $\mathcal{L}$ is a completely continuous operator.*

4 Problem (1.1), (1.2)

Theorem 4.1. *Let* (H_1)–(H_3) *hold. Then problem* (1.1), (1.2) *has at least one solution.*

Proof. By Lemma 3.1, we need to prove that $\mathcal{L}$ has a fixed point. Since $\mathcal{L}$ is completely continuous by Lemma 3.2, the Schaefer fixed point theorem guarantees the existence of a fixed point of $\mathcal{L}$ if the set $\mathcal{U} = \{(x,c) \in X : \ (x,c) = \lambda\mathcal{L}(x,c) \text{ for some } \lambda \in (0,1)\}$ is bounded. We show that $\mathcal{U}$ is bounded. $\square$

Example 4.2. Let $\phi = \phi_p$, $p > 1$, $\mu \in (0, p-1)$, $r, m, k \in C(J)$ and $f(t,x) = k(t) + |x|^\mu \arctan x$. Then conditions (H_1) and (H_2) are satisfied for $a = |r|$, $b = |m|$, $H = \max\{\pi/4, \sqrt[\mu]{\|k\|}\}$ and $D = -H$. Since $\phi^{-1} = \phi_q$, $q = p/(p-1)$, condition (H_3) is fulfilled for $w(v) = \|k\| + \pi v^\mu/2$. Theorem 4.1 guarantees that the problem

$$
\begin{gathered}
{}^c\mathcal{D}^\alpha \phi_p\Big({}^c\mathcal{D}^\beta x - |r(t)|\,{}^c\mathcal{D}^{\gamma_1} x - |m(t)|\,{}^c\mathcal{D}^{\gamma_2} x\Big) = k(t) + |x|^\mu \arctan x, \\
x(0) = x(T), \quad {}^c\mathcal{D}^\beta x(t)\big|_{t=0} = 0,
\end{gathered}
$$

has a solution.

References

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