On One Class of Solutions of Linear Matrix Equations

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Let

$$G(\varepsilon_0)=\big\{t,\varepsilon:\ t\in\mathbf{R},\ \varepsilon\in(\mathbf{0},\varepsilon_{\mathbf{0}}),\ \varepsilon_{\mathbf{0}}\in\mathbf{R}^+\big\}.$$

Definition 1. We say that a function $f(t,\varepsilon)$ belongs to the class $S(m;\varepsilon_0)$, $m\in\mathbf{N}\cup\{\mathbf{0}\}$, if:

1) $f:G(\varepsilon_0)\to\mathbf{C}$,

2) $f(t,\varepsilon)\in C^m(G(\varepsilon_0))$ at t,

3) $d^kf(t,\varepsilon)/dt^k=\varepsilon^kf_k(t,\varepsilon)$ $(0\le k\le m)$,

$$\|f\|_{S(m;\varepsilon_0)}\overset{def}{=}\sum_{k=0}^{m}\sup_{G(\varepsilon_0)}|f_k(t,\varepsilon)|<+\infty.$$

Definition 2. We say that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$, if

$$f(t,\varepsilon,\theta(t,\varepsilon))=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)),$$

and

1) $f_n(t,\varepsilon)\in S(m,\varepsilon_0)$ $(n\in\mathbf{Z})$,

2)

$$\|f\|_{F(m;\varepsilon_0;\theta)}\overset{def}{=}\sum_{n=-\infty}^{\infty}\|f_n\|_{S(m;\varepsilon_0)}<+\infty,$$

3) $\theta(t,\varepsilon)=\int\limits_0^t\varphi(\tau,\varepsilon)\,d\tau$, $\varphi\in\mathbf{R}^+$, $\varphi\in S(m,\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

Definition 3. We say that a matrix $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ belongs to the class $S_2(m;\varepsilon_0)$ $(m\in\mathbf{N}\cup\{0\})$, if $a_{jk}\in S(m;\varepsilon_0)$ $(j,k=\overline{1,N})$.

We define the norm

$$\|A(t,\varepsilon)\|_{S_2(m;\varepsilon_0)}=\max_{1\le j\le N}\sum_{k=1}^{N}\|a_{jk}(t,\varepsilon)\|_{S(m;\varepsilon_0)}.$$

Definition 4. We say that a matrix $B(t,\varepsilon,\theta)=(b_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ belongs to the class $F_2(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$, if $b_{jk}(t,\varepsilon,\theta)\in F(m;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$.

We define the norm

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} = \max_{1\le j\le N}\sum_{k=1}^{N}\|b_{jk}(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}. \tag{1}$$

Consider the linear non-homogeneous matrix equation

$$\frac{dX}{dt} = A(t,\varepsilon)X - XB(t,\varepsilon) + F(t,\varepsilon,\theta), \tag{2}$$

$A(t,\varepsilon), B(t,\varepsilon)\in S_2(m;\varepsilon_0)$, $F(t,\varepsilon,\theta)\in F(m;\varepsilon_0;\theta)$.

We study the existence of particular solutions of equation (2) in the class $F(m_1;\varepsilon_1;\theta)$ ($m_1\le m$, $\varepsilon_1\le\varepsilon_0$).

Lemma. *Let*

$$\frac{dx}{dt} = \lambda(t,\varepsilon)x + u(t,\varepsilon,\theta(t,\varepsilon)) \tag{3}$$

be a given scalar linear non-homogeneous first-order differential equation, where $\lambda(t,\varepsilon)\in S(m;\varepsilon)$, $\inf\limits_{G(\varepsilon_0)}|\operatorname{Re}\lambda(t,\varepsilon)| = \gamma > 0$, *and* $u(t,\varepsilon,\theta)\in F(m;\varepsilon_0;\theta)$. *Then equation* (3) *has a unique particular solution* $x(t,\varepsilon,\theta)\in F(m;\varepsilon_0;\theta)$. *This solution is given by the formula*

$$x(t,\varepsilon,\theta(t,\varepsilon)) = \int_T^t u(\tau,\varepsilon,\theta(\tau,\varepsilon))\exp\bigg(\int_\tau^t \lambda(s,\varepsilon)\,ds\bigg)\,d\tau,$$

where

$$T = \begin{cases} -\infty & \text{if } \operatorname{Re}\lambda(t,\varepsilon)\le -\gamma < 0,\\ +\infty & \text{if } \operatorname{Re}\lambda(t,\varepsilon)\ge \gamma > 0.\end{cases}$$

Moreover, there exists $K_0\in(0,+\infty)$ *such that*

$$\|x(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)} \le K_0\|u(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

Theorem 1. *Let equation* (2) *satisfy the next conditions:*

1) *there exist matrices* $L_1(t,\varepsilon), L_2(t,\varepsilon)\in S_2(m;\varepsilon_0)$ *such that*

    (a) $\det L_1(t,\varepsilon)\ge a_0>0$, $\det L_2(t,\varepsilon)\ge a_0>0$;

    (b) $L_1^{-1}(t,\varepsilon)A(t,\varepsilon)L_1(t,\varepsilon) = D_1(t,\varepsilon) = (d^1_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$,

    (c) $L_2(t,\varepsilon)B(t,\varepsilon)L_2^{-1}(t,\varepsilon) = D_2(t,\varepsilon) = (d^2_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$,

    where D_1, D_2 *are lower triangular matrices belonging to the class* $S_2(m;\varepsilon_0)$;

2) $\inf\limits_{G(\varepsilon_0)}|\operatorname{Re}(d^1_{jj}(t,\varepsilon) - d^2_{kk}(t,\varepsilon))|\ge b_0>0$ $(j,k=\overline{1,N})$.

Then there exists $\varepsilon_1\in(0,\varepsilon_0)$ *such that for all* $\varepsilon\in(0,\varepsilon_1)$ *there exists unique particular solution* $X(t,\varepsilon,\theta)\in F_2(m-1;\varepsilon_1;\theta)$ *of the matrix equation* (2).

Proof. We make in equation (2) the substitution

$$X = L_1(t,\varepsilon)Y(t,\varepsilon)L_2(t,\varepsilon), \tag{4}$$

where Y – the new unknown matrix. We obtain

$$\frac{dY}{dt}=\Big(D_1(t,\varepsilon)-L_1^{-1}(t,\varepsilon)\,\frac{dL_1(t,\varepsilon)}{dt}\Big)Y \\ -Y\Big(D_2(t,\varepsilon)+\frac{dL_2(t,\varepsilon)}{dt}\,L_2^{-1}(t,\varepsilon)\Big)+L_1^{-1}(t,\varepsilon)F(t,\varepsilon,\theta)L_2^{-1}(t,\varepsilon). \quad (5)$$

We denote

$$L_1^{-1}(t,\varepsilon)\,\frac{dL_1(t,\varepsilon)}{dt}=\varepsilon H_1\,\frac{dL_1(t,\varepsilon)}{dt}\,, \quad \frac{dL_2(t,\varepsilon)}{dt}\,L_2^{-1}(t,\varepsilon)=\varepsilon H_2\,\frac{dL_2(t,\varepsilon)}{dt}\,L_2^{-1}(t,\varepsilon), \\ L_1^{-1}(t,\varepsilon)F(t,\varepsilon,\theta)L_2^{-1}(t,\varepsilon)=F_1(t,\varepsilon,\theta).$$

Then equation (5) may be written as

$$\frac{dY}{dt}=D_1(t,\varepsilon)Y-YD_2(t,\varepsilon)-\varepsilon H_1(t,\varepsilon)Y-\varepsilon YH_2(t,\varepsilon)+F_1(t,\varepsilon,\theta).$$

By virtue Lemma and condition 2) of the theorem, the equation

$$\frac{dY_0}{dt}=D_1(t,\varepsilon)Y_0-Y_0D_2(t,\varepsilon)+F_1(t,\varepsilon,\theta)$$

has a unique solution $Y_0(t,\varepsilon,\theta)$ of the class $F(m;\varepsilon_0;\theta)$, and there exists $K_1\in(0,+\infty)$ such that

$$\|Y_0(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}\le K_1\|F_1(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

We construct the process of successive approximations, defining the initial approximation $Y_0(t,\varepsilon,\theta)$ and subsequent approximations defining as the solutions of the class $F(m-1;\varepsilon_0;\theta)$ of the equations

$$\frac{dY_k}{dt}=D_1(t,\varepsilon)Y_k-Y_kD_2(t,\varepsilon)-\varepsilon H_1(t,\varepsilon)Y_{k-1}-\varepsilon Y_{k-1}H_2(t,\varepsilon)+F_1(t,\varepsilon,\theta). \quad (6)$$

Using the ordinary technique of the contraction mapping principle it is easy to show that there exists $\varepsilon_1\in(0,\varepsilon_0)$ such that for all $\varepsilon\in(0,\varepsilon_1)$ process (6) convergence by the norm (1) to the solution of the class $F(m-1;\varepsilon_1;\theta)$ of equation (2).