Some Properties of the Lyapunov, Perron, and Upper-Limit Stabilities of Differential Systems

I. N. Sergeev

Lomonosov Moscow State University, Moscow, Russia
E-mail: igniserg@gmail.com

This report is a logical continuation of the previous report [15] by the same author.

For a given zero neighborhood G in the Euclidean space $\mathbb{R}^n$, we consider the system

$$\dot{x} = f(t,x), \ \ x \in G, \ \ f(t,0) = 0, \ \ t \in \mathbb{R}_+ \equiv [0,+\infty), \tag{1}$$

where the right-hand side satisfies the condition $f, f'_x \in C(\mathbb{R}_+ \times G)$ and the zero solution is allowed. Let $S_*(f)$ and $S_\delta(f)$ denote the set of all non-continuable solutions x of system (1), given by the initial conditions $|x(0)| \neq 0$ and $0 < |x(0)| < \delta$, respectively.

Definition 1. We say that system (1) (more precisely, its zero solution) possesses the following *upper-limit* property:

1) *stability* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution $x \in S_\delta(f)$ satisfies the condition
$$\varlimsup_{t\to+\infty} |x(t)| < \varepsilon; \tag{2}$$

2) *partial stability* if for any $\varepsilon, \delta > 0$ at least one solution $x \in S_\delta(f)$ satisfies condition (2);

3) *asymptotic stability* if there exists $\delta > 0$ such that any solution $x \in S_\delta(f)$ satisfies the condition
$$\varlimsup_{t\to+\infty} |x(t)| = 0 \quad \Big(\iff \lim_{t\to+\infty} |x(t)| = 0 \Big); \tag{3}$$

4) *global stability* if all solutions $x \in S_*(f)$ satisfy condition (3);

5) *instability* if there is no upper-limit stability, that is there is an $\varepsilon > 0$ such that for any $\delta > 0$ at least one solution $x \in S_\delta(f)$ does not satisfy condition (2) (in particular, it is not defined on the whole semi-axis $\mathbb{R}_+$);

6) *complete instability* if there is no upper-limit partial stability, that is there are $\varepsilon, \delta > 0$ such that no solution $x \in S_\delta(f)$ satisfies condition (2);

7) *asymptotic instability* if there is no upper-limit asymptotic stability, that is for any $\delta > 0$ at least one solution $x \in S_\delta(f)$ does not satisfy condition (3);

8) *total instability* if for some $\varepsilon > 0$ no solution $x \in S_*(f)$ satisfies condition (2).

Definition 2. Analogously to Definition 1, we say that system (1) possesses the corresponding *Perron* property (*stability*, *partial stability*, *asymptotic stability*, *global stability*, *instability*, *complete instability*, *asymptotic instability*, *total instability*) if it has property 1)–8) (respectively) from Definition 1, in which the upper limits under conditions (2) and (3) are replaced by the lower limits everywhere. Similarly, system (1) possesses the corresponding *Lyapunov* property if it has respectively property 1)–8) from Definition 1, in which:

(a) the upper limit at $t \to +\infty$ in condition (2) is replaced everywhere by an exact upper bound over all $t \in \mathbb{R}_+$;

(b) the requirement of Lyapunov stability is added to asymptotic and global stabilities in properties 3 and 4, respectively;

(c) property (7) is replaced by negation of the resulting property (3), that is either for any $\delta > 0$ at least one solution $x \in S_\delta(f)$ does not satisfy condition (3) or there is no Lyapunov stability.

We will be especially interested in particular cases of n-dimensional system (1): *one-dimensional* $(n = 1)$ and *two-dimensional* $(n = 2)$ systems, *autonomous* system

$$\dot{x} = f(x), \quad f(0) = 0, \quad x \in G \subset \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{4}$$

with right-hand side $f \in C^1(G)$, and *linear* system

$$\dot{x} = A(t)x, \quad x \in G \equiv \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{5}$$

defined by its continuous operator function $A : \mathbb{R}_+ \to \operatorname{End} \mathbb{R}^n$.

According to the next two theorems, the Lyapunov complete and total instabilities are equivalent, but this statement does not carry over the Perron instabilities and the upper-limit ones.

Theorem 1. *If system* (1) *is Lyapunov completely unstable, then it is Lyapunov totally unstable too.*

Theorem 2. *There exists a two-dimensional system* (1)*, which is Perron and upper-limit completely unstable, but neither Perron nor upper-limit totally unstable; moreover, it has at least one solution* $x \in S_*(f)$ *satisfying condition* (3)*.*

It can be seen from the following two theorems that, in the linear case, the assertion of Theorem 1 extends also to the complete and total instability of both Perron and upper-limit types, as well as to asymptotic and global stability of any type at all.

Theorem 3. *If the linear system* (5) *is Lyapunov, or Perron, or upper-limit completely unstable, then it is respectively Lyapunov, or Perron, or upper-limit totally unstable too.*

Theorem 4. *If the linear system* (5) *is Lyapunov, or Perron, or upper-limit asymptotically stable, then it is respectively Lyapunov, or Perron, or upper-limit globally stable too.*

In the autonomous case, Theorem 1 can be significantly reinforced, which is what the following two theorems do.

Theorem 5. *If for the autonomous system* (4) *at least one of the following six properties is satisfied: the Perron, Lyapunov, or upper-limit complete or total instability, then the other five of them are also satisfied.*

Theorem 6. *If the autonomous system* (4) *is not, at least, Lyapunov, or Perron, or upper-limit totally unstable, then it is both Lyapunov, and Perron, and upper-limit partially stable.*

Each of the upper-limit properties occupies a logically intermediate position between its Lyapunov and Perron analogs.

According to the following two theorems, in the one-dimensional and in the linear cases, all the upper-limit properties are indistinguishable from the corresponding Lyapunov properties, and under the additional condition of autonomy, also from the Perron ones.

Theorem 7. *For any one-dimensional system* (1)*, each of its upper-limit properties is equivalent to the analogous Lyapunov property, and in the case of an autonomous one-dimensional system, it is equivalent to the Perron property too.*

Theorem 8. *For any linear system* (5)*, each of its upper-limit properties is equivalent to the analogous Lyapunov property, and in the case of autonomic linear system, it is equivalent to the Perron property too.*

Already in the linear case, the upper-limit properties, although they coincide with the Lyapunov ones, can be directly opposite to the Perron ones.

Theorem 9. *For each $n \in \mathbb{N}$, there exists a linear n-dimensional system* (1)*, which is globally Perron stable, but both Lyapunov and upper-limit totally unstable.*

If the system is not linear and not one-dimensional, then even in the autonomous case, the upper-limit properties can also sharply contrast with both the Lyapunov and Perron ones.

Theorem 10. *There exists a two-dimensional autonomous system* (4)*, which is Lyapunov unstable, but both Perron and upper-limit globally stable.*

Theorem 11. *There exists a two-dimensional autonomous system* (4)*, which is globally Perron stable, but both Lyapunov and upper-limit unstable.*

Theorems 10 and 11 cannot be strengthened by replacing instability in them with complete instability (and even more so with total instability, this would contradict Theorem 5). The next theorem serves as a certain modification of Theorem 11.

Theorem 12. *There exists a two-dimensional autonomous system* (4)*, which is Perron stable, but both Lyapunov and upper-limit unstable; moreover, all its fixed points fill some ray $C \subset G$ with origin at zero, and any of its solutions $x \in S_*(f)$ with initial values $x(0) \notin C$ satisfies the relations*

$$0 = \varliminf_{t\to+\infty} |x(t)| < \varlimsup_{t\to\infty} |x(t)| = +\infty.$$

The system from Theorem 10 is described in [2, p. 6.3], and its simplified version is in [3, § 18].

For more information on these issues, see the reports [1, 4–6, 9–12, 14, 17–19]. The proofs of the above Theorems 1–12 are mainly contained in the papers [7, 8, 13, 16, 20].

References

[1] A. A. Bondarev, One example of an unstable system. (Russian) *Differ. Uravn* **55** (2019), no. 6, 899.

[2] B. F. Bylov, R. È. Vinograd, D. M. Grobman and V. V. Nemyckiĭ, *Theory of Ljapunov Exponents and its Application to Problems of Stability.* (Russian) Izdat. "Nauka", Moscow, 1966.

[3] A. F. Filippov, *Introduction to the Theory of Differential Equations.* (Russian) Editorial URSS, Moscow, 2004.

[4] I. N. Sergeev, Determination of Perron stability and its relationship with Lyapunov stability. (Russian) *Differ. Uravn.* **54** (2018), no. 6, 855–856.

[5] I. N. Sergeev, On the study on the Perron stability of one-dimensional and autonomous differential systems. *Differ. Uravn.* **54** (2018), no. 11, 1561–1562.

[6] I. N. Sergeev, Study of the Perron stability of linear differential systems. (Russian) *Differ. Uravn.* **54** (2018), no. 11, 1571–1572.

[7] I. N. Sergeev, Perron stability and its study at the first approximation. (Russian) *Dokl. Akad. Nauk, Ross. Akad. Nauk* **486** (2019), no. 1, 20–23; translation in *Dokl. Math.* **99** (2019), no. 3, 252–254.

[8] I. N. Sergeev, Definition and some properties of Perron stability. (Russian) *Differ. Uravn.* **55** (2019), no. 5, 636–646; translation in *Differ. Equ.* **55** (2019), no. 5, 620–630.

[9] I. N. Sergeev, On the study of Perron and Lyapunov stability properties in the first approximation. *Differ. Uravn* **55** (2019), no. 6, 897–899.

[10] I. N. Sergeev, Lyapunov and Perron stability of solutions of differential systems. (Russian) *Abstracts of the International Conference Dedicated to the 80th Anniversary of Academician of the Russian Academy of Sciences V. A. Sadovnichy "Modern Problems of Mathematics and Mechanics"*, pp. 358–361, MAKS Press, Moscow, 2019.

[11] I. N. Sergeev, On the Perron and Lyapunov stability of solutions. (Russian) *Abstracts of XIX International Scientific Conference on Differential Equations "Erugin readings – 2019", part 1*, pp. 52–57, Inst. of Mathematics of the National Acad. of Sciences of Belarus, Minsk, 2019.

[12] I. N. Sergeev, Perron and Lyapunov stability properties and their investigation by the first approximation. (Russian) *Abstracts of III International Scientific Conference "Autumn mathematical readings in Adygea"*, pp. 83–86, Adyghe State University, Maikop, 2019.

[13] I. N. Sergeev, Dependence and independence of the Perron and Lyapunov stability properties on the system phase domain. (Russian) *Differ. Uravn.* **55** (2019), no. 10, 1338–1346; translation in *Differ. Equ.* **55** (2019), no. 10, 1294–1303.

[14] I. N. Sergeev, On the dependence and independence of the Perron and Lyapunov stability properties on the phase domain of the system. (Russian) *Differ. Uravn* **55** (2019), no. 11, 1572–1573.

[15] I. N. Sergeev, Definition and properties of perron stability of differential systems. *Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2019*, Tbilisi, Georgia, December 7–9, 2019, pp. 162–166; `http://www.rmi.ge/eng/QUALITDE-2019/Sergeev_workshop_2019.pdf`.

[16] I. N. Sergeev, Studying the Perron and Lyapunov stability properties by the first approximation. (Russian) *Differ. Uravn.* **56** (2020), no. 1, 84–93; translation in *Differ. Equ.* **56** (2020), no. 1, 83–82.

[17] I. N. Sergeev, Indistinguishability of some Lyapunov, Perron, and upper-limit stability properties of differential systems. (Russian) *Abstracts of All-Russian Conference with International Participation Dedicated to Prof. N. V. Azbelev and Prof. E. L. Tonkov, "Control Theory and Mathematical Modelling"*, pp. 118–120, Udmurt State University, Izhevsk, 2020.

[18] I. N. Sergeev, Some features of the Perron and Lyapunov stability properties of autonomous systems. (Russian) *Differ. Uravn* **56** (2020), no. 6, 830–831.

[19] I. N. Sergeev, Determination of upper-limit stability and its relationship with Lyapunov and Perron stability. (Russian) *Differ. Uravn* **56** (2020), no. 11, 1556–1557.

[20] I. N. Sergeev, Lyapunov, Perron and upper-limit stability properties of autonomous differential systems. (Russian) *Izvestiya Instituta Matematiki i Informatiki Udmurtskogo Gosudarstvennogo Universiteta* **55** (2020) (in press).