On Sturm-Type Theorems for Nonlinear Equations

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1 Introduction

In this paper we consider the nonlinear equation of higher ($n > 2$) order:

$$y^{(n)} + p(t, y, y', \dots, y^{n-1})|y|^k \operatorname{sgn} y = 0, \;\; k \in (0, 1) \cup (1, \infty), \tag{1.1}$$

where, for some $m, M \in \mathbb{R}$, the inequalities $0 < m \leq |p(t, \xi_1, \xi_2, \dots, \xi_n)| \leq M < \infty$ hold, the function $p(t, \xi_1, \xi_2, \dots, \xi_n)$ is continuous and Lipschitz continuous in $(\xi_1, \xi_2, \dots, \xi_n)$.

We study some oscillatory properties of (1.1) and compare them, for different n, with oscillatory properties of the linear equation

$$y^{(n)} + p(t)y = 0. \tag{1.2}$$

We obtain Sturm-type theorems.

2 Sturm's and Kondratiev's theorems

We know the classical result on properties of zeros of solutions to a second order linear equation, called Sturm's theorem.

Theorem 2.1 (Sturm J. Ch. F.)**.** *Consider two linearly independent solutions to the equation*

$$y'' + Q(t)y = 0$$

with a continuous function $Q(t)$, and let one of the solutions have two consecutive zeros. Then there is exactly one zero of another solution between those consecutive zeros.

This result was generalized by V. A. Kondratiev for linear equations of higher order.

Theorem 2.2 (V. A. Kondratiev, 1959)**.** *Suppose that solution to equation*

$$y''' + p(t)y = 0,$$

where $p(t)$ is continuous and $p(t) > 0$ for every t (or $p(t) < 0$ for every t), has consecutive zeros x_1 and x_2. Then every other solution to the equation has no more than two zeros on $[x_1, x_2]$.

Theorem 2.3 (V. A. Kondratiev, 1959)**.** *Suppose that solution to equation*

$$y^{IV} + q(t)y = 0,$$

where $p(t)$ is continuous and $p(t) > 0$ for every t, has consecutive zeros x_1 and x_2. Then every other solution to the equation has no more than four zeros on $[x_1, x_2]$.

Theorem 2.4 (V. A. Kondratiev, 1959)**.** *Suppose that solution to equation*

$$y^{IV} + q(t)y = 0,$$

where $p(t)$ is continuous and $p(t) < 0$ for every t, has consecutive zeros x_1 and x_2. Then every other solution to the equation has no more than three zeros on $[x_1, x_2]$.

And for linear equations of fifth and higher order V. A. Kondratiev proved (1961) that for $n \geq 5$ and $p(t) \geq 0$ there exists a solution to

$$y^{(n)} + p(t)y = 0$$

with arbitrary number of zeros between two consecutive zeros of another solution (see [5, 6]).

3 Theorem for the nonlinear equation

We consider equation (1.1) to be a generalisation of equation (1.2). Equation (1.1), which is, in turn, a generalisation to Emden equation, was studied in the [1–4, 7–13], and from variety of results obtained, we derive a theorem that serves as an analogue of Sturm's and Kondratiev's theorems, but for the nonlinear equation.

Theorem 3.1. *Suppose that a solution to equation* (1.1)*, where $p(t, \xi_1, \xi_2, \dots, \xi_n) > 0$ (or n is odd and $p(t, \xi_1, \xi_2, \dots, \xi_n) < 0$), has consecutive zeros x_1 and x_2. If $k \in (0, 1) \cup (1, +\infty)$, then there exists a solution to* (1.1) *with arbitrary finite number of zeros on $[x_1, x_2]$. If $k \in (0, 1)$, then there exists a solution to* (1.1) *with countable set of zeros on $[x_1, x_2]$, and a solution with a set of zeros on $[x_1, x_2]$ with the cardinality of the continuum.*

For the nonlinear equation results are the same for every $n > 2$, unlike results for linear equations. Any number of zeros is possible, irregardless of n.

Remark 1. Equation with even order n and negative $p(t, \xi_1, \xi_2, \dots, \xi_n)$ require more research, although we already know that we can't expect same results. As [1, Chapter 7] shows, solutions even to $y^{(4)} - y^3 = 0$ and $y^{(4)} + y^3 = 0$ differ greatly in their behaviour.

Remark 2. When $k = 1$, equation (1.1), in general, is not linear, but as we get a linear equation as special case when $p(t, \xi_1, \xi_2, \dots, \xi_n)$ depends only on t, in general case we can expect properties, similar to Sturm's and Kondratiev's results, where possible number of zeros depends on n. Further research is required.

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