Baire and Local Baire Classes of Functionals on the Space of Linear Differential Systems

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For a given $n \in \mathcal{N}$ let us consider the set $\mathcal{M}^n$ of linear systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

with continuous and bounded matrix-valued functions $A : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$, which we identify with the corresponding linear systems.

In the contemporary theory of Lyapunov exponents the set $\mathcal{M}^n$ is usually equipped with the *uniform* and *compact-open* topologies defined respectively by the metrics

$$\rho_U(A, B) = \sup_{t \in \mathbb{R}_+} \|A(t) - B(t)\| \ \text{ and } \ \rho_C(A, B) = \sup_{t \in \mathbb{R}_+} \min\big\{\|A(t) - B(t)\|, 2^{-t}\big\},$$

with $\|\cdot\|$ being a matrix norm (e.g., the spectral one). The resulting topological spaces will be denoted by $\mathcal{M}^n_U$ and $\mathcal{M}^n_C$. V. M. Millionshchikov [5, 6] proposed using the Baire classification of discontinuous functions [2] to describe the dependence of various characteristics of asymptotic behavior of solutions to linear differential systems on their coefficients.

To recall what the Baire classification is, it is convenient to introduce the following notation. Let M be a metric space and F a collection of functions $f : M \to \mathbb{R}$. For each $k \in \mathcal{N}_0 \equiv \mathcal{N} \sqcup \{0\}$ we define the function class $[F]_k$ by induction as follows:

1. The class $[F]_0$ coincides with the class F.
2. The class $[F]_k$ consists of functions $f : M \to \mathbb{R}$ that can be represented in the form

   $$f(x) = \lim_{j \to \infty} f_j(x), \ \ x \in M,$$

   where functions $f_j : M \to \mathbb{R}$, $j \in \mathcal{N}$, belong to the class $[F]_{k-1}$.

Definition 1. Let M be a metric space. For each number $k \in \mathcal{N}$ we define the k-th Baire class $\mathfrak{F}_k(M)$ by the equality $\mathfrak{F}_k(M) = [C(M)]_k$, where $C(M)$ is the set of all continuous functions $f : M \to \mathbb{R}$. Besides, for each $k \in \mathcal{N}$ we define the exact k-th Baire class $\check{\mathfrak{F}}_k(M)$ by the equality $\check{\mathfrak{F}}_k(M) = \mathfrak{F}_k(M) \setminus \mathfrak{F}_{k-1}(M)$.

Put simply, Definition 1 tells that a function f belongs to the k-th Baire class if there exists a k-indexed sequence of continuous functions such that f can be obtained by taking pointwise limits of this sequence for k times (as each of its indices successively tends to infinity). Thus, when considering functionals on the space of linear differential systems, it is natural from the viewpoint of applications to require that the values of the prelimit functionals can be calculated from information about the system on finite intervals of the time semiaxis. This argument leads to the following [7]

Definition 2. We say that a functional $\varphi : \mathcal{M}^n \to \mathbb{R}$ has a compact support if there exists a $T > 0$ such that $\varphi(A) = \varphi(B)$ whenever $A, B \in \mathcal{M}^n$ coincide on the interval $[0, T]$. The set of all functionals with compact support will be denoted by $\mathfrak{C}^n$.

Definition 3. The k-th class of formulas $\mathfrak{C}^n_k$ is defined by

$$\mathfrak{C}^n_k = \big[\mathfrak{F}_0(\mathcal{M}^n_C) \cap \mathfrak{C}^n\big]_k, \ \ k \in \mathcal{N}_0.$$

It was established in the paper [5] that the classes $\mathfrak{C}^n_k$ and $\mathfrak{F}_k(\mathcal{M}^n_C)$ coincide with each other for all $k \geq 1$. This fact emphasizes the importance of studying the compact-open topology on the space $\mathcal{M}^n$.

For a more detailed classification let us give the following

Definition 4. The (k, m)-th class of formulas $\mathfrak{C}^n_{k,m}$ is defined by

$$\mathfrak{C}^n_{k,m} = \big[\mathfrak{F}_k(\mathcal{M}^n_C) \cap \mathfrak{C}^n\big]_m, \ \ k, m \in \mathcal{N}_0.$$

It is of interest to consider inclusions between different classes of formulas just introduced. The inclusion criterion is provided by the following

Theorem 1. *Let $i, j, k, m \in \mathcal{N}_0$. The inclusion $\mathfrak{C}^n_{i,j} \subset \mathfrak{C}^n_{k,m}$ holds if and only if $j \leq k$ and $i + j \leq k + m$.*

The main subject of study in the theory of Lyapunov exponents is not arbitrary functionals on the space of linear systems but rather the invariants of action of various transformation groups, particularly, Lyapunov transformations [1, Ch. III, § 1]. Hence, we give the following

Definition 5. A continuously differentiable matrix-valued function $L : \mathbb{R}_+ \to R^{n\times n}$ is called a Lyapunov transformation if $L(t)$ is invertible for all $t \in \mathbb{R}_+$ and the condition

$$\sup_{t\geq 0} \big(\|L(t)\| + \|L^{-1}(t)\| + \|\dot{L}(t)\|\big) < \infty$$

is satisfied. Systems $A, B \in \mathcal{M}^n$ are said to be Lyapunov equivalent if there exists a Lyapunov transformation that reduces the system A into the system B. A functional $\varphi : \mathcal{M}^n \to \mathbb{R}$ is called a Lyapunov invariant if $\varphi(A) = \varphi(B)$ whenever A and B are Lyapunov equivalent. The set of all Lyapunov invariants will be denoted by $\mathfrak{L}^n$.

It was shown in the paper [4] that the set $[\mathfrak{C}^n]_1 \cap \mathfrak{L}^n$ contains only constants and that $[\mathfrak{C}^n]_k \not\supset \mathfrak{F}_{k+1}(\mathcal{M}^n_C) \cap \mathfrak{L}^n$. These statements are supplemented by the following

Theorem 2. *Let i, j, k, m be nonnegative integers and $j \geq 2$. In order for the inclusion $\mathfrak{C}^n_{i,j} \cap \mathfrak{L}^n \subset \mathfrak{C}^n_{k,m} \cap \mathfrak{L}^n$ to hold it is necessary that $m \geq j$ and sufficient that $m \geq j + 1$.*

Remark. In the case $j \geq 2$ it is unknown to the author whether the classes $\mathfrak{C}^n_{i,j} \cap \mathfrak{L}^n$ and $\mathfrak{C}^n_{k,j} \cap \mathfrak{L}^n$ coincide for different i and k.

Since continuity (or discontinuity) of a function varies from point to point in its domain, a local Baire classification of functions makes sense [8].

Definition 6. Let M be a metric space and $k \in \mathcal{N}_0$. We say that a function f belongs to the k-th Baire class at a point $x_0 \in M$ and write $f \in \mathfrak{F}_k(\mathcal{M}^n_C, x_0)$ if there exists a neighborhood U of the point x_0 such that the restriction of f to U belongs to the k-th Baire class. If, in addition, $f \notin \mathfrak{F}_{k-1}(\mathcal{M}^n_C, x_0)$, then we say that a function f belongs to the exact k-th Baire class at the point $x_0 \in M$ and write $f \in \check{\mathfrak{F}}_k(\mathcal{M}^n_C, x_0)$. If $f \in \check{\mathfrak{F}}_k(\mathcal{M}^n_C, x_0)$ for all $x_0 \in M$, the function f is said to be uniform of the k-th Baire class.

It is well known [8] that each Lyapunov exponent considered as a function on $\mathcal{M}^n_C$ is uniform (of the second Baire class).

It turns out that this property is shared by all Lyapunov invariants as shown by the following

Theorem 3. *Each Lyapunov invariant $\mathcal{M}^n_C \to \mathbb{R}$ that belongs to a certain exact Baire class is uniform of that class.*

By contrast, each Lyapunov exponent considered as a function on $\mathcal{M}^n_U$ belongs to the zeroth Baire class at some points and to the second Baire class at others. It is known that for the two lowest exponents there are no points at which either of them belongs to the exact first Baire class [3, 8]. For the rest exponents it is conjectured but not proved to date.

The question naturally arises which local Baire classes can a general Lyapunov invariant $\mathcal{M}^n_U \to \mathbb{R}$ belong to at different points?

Theorem 4. *For every $n \in \mathcal{N}$ there exist a Lyapunov invariant $\varphi : \mathcal{M}^n_U \to [0,1]$ and a set of points $\{A_i \in \mathcal{M}^n : \ i \in \mathcal{N}\}$ such that*

$$\varphi \in \bigcap_{i\in\mathcal{N}} \check{\mathfrak{F}}_i(\mathcal{M}^n_U, A_i).$$

Theorem 5. *For any integers $n \geq 1$ and $N \geq 2$ there exist a Lyapunov invariant $\varphi : \mathcal{M}^n \to [0,1]$ and a set of points $\{A_i \in \mathcal{M}^n : \ i = 1, \dots, N\}$ such that*

$$\varphi \in \bigcap_{i=1}^{N} \check{\mathfrak{F}}_i(\mathcal{M}^n_U, A_i) \cap \check{\mathfrak{F}}_N(\mathcal{M}^n_C).$$

The author expresses a deep gratitude to V. V. Bykov for posing the problem and for attention to the research.

References

[1] L. Ya. Adrianova, *Introduction to Linear Systems of Differential Equations*. Translated from the Russian by Peter Zhevandrov. Translations of Mathematical Monographs, 146. American Mathematical Society, Providence, RI, 1995.

[2] R. Baire and A. Denjoy, *Leçons sur les Fonctions Discontinues: Professées au Collège de France*. Gauthier-Villars, 1905.

[3] V. V. Bykov, Local Baire classification of Lyapunov exponents. *Trudy Seminara im. I. G. Petrovskogo (Petrovskii Seminars)* No. 26 (2007); translation in *J. Math. Sci. (N.Y.)* **143** (2007), no. 4, 3217–3225.

[4] V. V. Bykov, On Baire classes of Lyapunov invariants. (Russian) *Mat. Sb.* **208** (2017), no. 5, 38–62; translation in *Sb. Math.* **208** (2017), no. 5-6, 620–643.

[5] V. M. Millionshchikov, Baire classes of functions and Lyapunov exponents. I. (Russian) *Differentsial'nye Uravneniya* **16** (1980), no. 8, 1408–1416; translation in *Differ. Equ.* **16** (1980), no. 8, 902–907.

[6] V. M. Millionshchikov, Lyapunov exponents as functions of a parameter. (Russian) *Math. USSR-Sb.* **65** (1990), no. 2, 369–384; translation in *Mat. Sb. (N.S.)* **137(179)** (1988), no. 3, 364–380.

[7] I. N. Sergeev, Baire classes of formulas for exponents of linear systems. (Russian) *Differ. Uravn.* **31** (1995), no. 12, 2092–2093.

[8] I. N. Sergeev, On local Baire classes of Lyapunov exponents. (Russian) *Differ. Uravn.* **32** (1996), no. 11, 1577.