Qualitative Behavior of Solutions of Impulsive Weakly Nonlinear Hyperbolic Equation

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1 Introduction

Beginning from the pioneering works of A. M. Samoilenko the theory of differential equations with impulses [1,6,9] provides a mathematical tools for describing the behavior of many evolutionary processes with instant changes. The important subclass of the systems with discontinuous trajectories are impulsive (discontinuous) dynamical systems, whose trajectories have jumps after intersection with a given surface M in the phase space [5]. In infinite-dimensional phase spaces the theory of global attractors is a basis for qualitative analysis of solutions [10]. The lack of continuous dependence on initial data in impulsive dynamical systems do not allow us to apply directly the methods of the theory of global attractors. Nevertheless, one can use more general concept of uniform attractor [8] in this case. By definition, uniform attractor is a compact uniformly attracting (w.r.t. bounded initial data) set which is minimal among all such sets. Existence of such a set in impulsive infinite dimensional case firstly was proved in [3] for weakly nonlinear parabolic equation. It turned out that in the case of infinitely many impulsive points along trajectories the uniform attractor Θ has non-empty intersection with impulsive set M. As a consequence, it is neither invariant no stable set w.r.t. the impulsive semi-flow. In the paper [4] a new stability concept was introduced basing on the properties of the set $\Theta \setminus M$. In the present paper we investigate stability of uniform attractor for the weakly nonlinear second order evolutionary problem with impulses.

2 Uniform attractors of impulsive semi-flows

Impulsive dynamical system on normed space E consists of continuous semigroup $V : R_+ \times E \to E$, impulsive set $M \subset E$ and impulsive map $I : M \to E$. The phase point moves along trajectories of V until the moment τ when the phase point $x(t)$ reaches the set M. At that moment the point instantaneously moves into a new position $Ix(\tau)$.

We need the following assumptions [5]:

$$\begin{gathered} M \text{ is closed}, \;\; M \cap IM = \varnothing, \\ \forall\, x \in M, \;\; \exists\, \tau = \tau(x) > 0, \;\; \forall\, t \in (0, \tau) \;\; V(t, x) \notin M, \\ \text{every impulsive trajectory is defined on } [0, +\infty). \end{gathered} \tag{2.1}$$

Let us introduce notations:

$$\forall\, x \in M \;\; Ix = x^+, \;\; \forall\, x \in E \quad M^+(x) = \Big(\bigcup_{t>0} V(t, x)\Big) \cap M.$$

If $M^+(x) \neq \varnothing$, then from the continuity of V we deduce that there exists a moment $s > 0$ such that

$$\forall\, t \in (0,s) \ \ V(t,x) \notin M, \ \ V(s,x) \in M.$$

Then the impulsive semiflow G is described by the following construction:

if for $x \in E$ and for every $t > 0$ $V(t,x) \notin M$, then

$$G(t,x) = V(t,x).$$

Otherwise,

$$G(t,x) = \begin{cases} V(t-t_n, x_n^+), & t \in [t_n, t_{n+1}), \\ x_{n+1}^+, & t = t_{n+1}, \end{cases} \tag{2.2}$$

where $t_0 = 0$, $t_{n+1} = \sum\limits_{k=0}^{n} s_k$, $x_{n+1}^+ = IV(s_n, x_n^+)$, $x_0^+ = x$, s_n is a moment of impulsive perturbation, which is characterized by inclusion $V(s_n, x_n^+) \in M$.

Formula (2.2) defines (not necessary continuous) semigroup $G : R_+ \times E \to E$, which is called impulsive semiflow.

We will use the following notations:

$$b(E) \ \text{ is a set of all bounded subsets of } E;$$

$$\operatorname{dist}(A,B) = \sup_{x \in A} \inf_{y \in B} \|x-y\|_E;$$

$$O_\delta(A) = \big\{x \in E : \ \operatorname{dist}(x,A) < \delta\big\}.$$

Definition 2.1 ([3])**.** A compact set $\Theta \subset E$ is called a uniform attractor of the impulsive semiflow G if

(1) Θ is uniformly attracting set, i.e.

$$\forall\, B \in b(E) \ \ \operatorname{dist}(G(t,B), \Theta) \to 0, \ \ t \to \infty;$$

(2) Θ is the minimal among all sets satisfying (1).

Theorem 2.1 ([3])**.** *Assume that the impulsive semiflow G is dissipative, i.e.,*

$$\exists\, B_0 \in b(E), \ \ \forall\, B \in b(E), \ \ \exists\, T = T(B), \ \ \forall\, t \geq T \ \ G(t,B) \subset B_0. \tag{2.3}$$

Then G has uniform attractor Θ if and only if G is asymptotically compact, i.e.,

$$\forall\, \{t_n \nearrow \infty\} \ \ \forall\, \{x_n\} \in b(E), \ \ \forall\, \{t_n \nearrow \infty\}, \ \ \forall\, \{x_n\} \in b(E) \ \ \{G(t_n, x_n)\} \ \textit{ is precompact in } E. \tag{2.4}$$

Moreover, the following equality takes place

$$\Theta = \omega(B_0) := \bigcap_{\tau > 0} \overline{\bigcup_{t \geq \tau} G(t, B_0)}. \tag{2.5}$$

Definition 2.2 ([2])**.** A set $A \subset E$ is called stable with respect to semiflow G if

$$A = D^+(A) := \bigcup_{x \in A} \big\{y : \ \ y = \lim G(t_n, x_n), \ x_n \to x, \ t_n \geq 0\big\}. \tag{2.6}$$

Remark. As $A \subset D^+(A)$, so (2.6) is equivalent to $D^+(A) \subset A$.

It is known that for continuous semiflows a uniform attractor is invariant and stable in the sense (2.6). Our main goal is to prove that for impulsive semiflow generated by impulsive perturbed wave equation the uniform attractor Θ satisfies the property

$$D^+(\Theta \setminus M) \subset \overline{\Theta \setminus M}. \tag{2.7}$$

3 Impulsive wave equation

We consider the triplet of Hilbert spaces $V \subset H \subset V^*$ with compact and dense embedding. Let $\|\cdot\|$ and $(\cdot\,,\,\cdot)$ be the norm and scalar product in H, $A : V \to V^*$ be a linear continuous self-adjoint coercive operator. The function $\langle Au, u\rangle^{\frac{1}{2}}$ defines a norm in the space V, which is denoted by $\|u\|_V$.

We consider the following evolutionary problem ($\beta > 0$):

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + 2\beta\,\dfrac{\partial y}{\partial t} + Ay = \varepsilon F(y), \\ y\big|_{t=0} = y_0 \in V, \\ y_t\big|_{t=0} = y_1 \in H, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is a small parameter, $F : H \mapsto H$ is a given Lipschitz continuous map. It is known [10] that in the phase space $E = V \times H$ this problem generates continuous semigroup $V : R_+ \times E \to E$, where

$$\text{for } \; z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in E, \;\; V(t, z_0) = z(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix}.$$

The norm in E is given by the equality:

$$\text{for } \; z = \begin{pmatrix} y \\ w \end{pmatrix} \in E, \;\; \|z\|_E = \|y\|_V + \|w\|.$$

Qualitative behavior of linear impulsive wave equation firstly was considered in [7]. It was shown that it is natural to consider an impulsive set as a level set of some seminorm l_p, where

$$\forall\, z \in E \;\; l_p(z) \to \|z\|_E, \;\; p \to \infty.$$

Let $\{\lambda_i\}, \{\psi_i\}$ be solutions of spectral problem:

$$\forall\, i \geq 1 \;\; A\psi_i = \lambda_i \psi_i, \;\; 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \;\; \lambda_i \to \infty, \;\; i \to \infty.$$

For $p \geq 1$ let us consider $l_p : E \to R$:

$$\text{for } \; z = \begin{pmatrix} y \\ w \end{pmatrix} \in E, \;\; l_p(z) = \Big(\sum_{i=1}^{p} \big\{ \lambda_i (y, \psi_i)^2 + (w, \psi_i)^2 \big\} \Big)^{\frac{1}{2}}.$$

For fixed $p \geq 1$, $a > 0$, $\mu > 0$ let us put

$$\begin{aligned} M &= \big\{ z \in E : \;\; l_p(z) = a \big\}, \\ M' &= \big\{ z \in E : \;\; l_p(z) = a(1+\mu) \big\}, \end{aligned} \tag{3.2}$$

$I : M \to M^{'}$ such that

$$\text{for } \; z = \sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M,$$

$$I(z) \in \Big\{ \sum_{i=1}^{p} \begin{pmatrix} c_i' \\ d_i' \end{pmatrix} \psi_i + \sum_{i=p+1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i : \;\; \sum_{i=1}^{p} \big\{ \lambda_i (c_i')^2 + (d_i')^2 \big\} = a^2 (1+\mu)^2 \Big\}. \tag{3.3}$$

The main result is the following theorem.

Theorem 3.1. *For every impulsive map $I : M \mapsto M'$ satisfying* (3.3) *and for sufficiently small $\varepsilon > 0$ the impulsive problem* (3.1)–(3.3) *generates impulsive semiflow $G : R_+ \times E \to E$, which has uniform attractor Θ and* (2.7) *takes place.*

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