The Ambrosetti–Prodi Problem for First Order Periodic ODEs with Sign-Indefinite Nonlinearities

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The Ambrosetti–Prodi problem for an equation of the form

$$F(x) = s \tag{1}$$

consists of determining how varying the parameter s affects the number of solutions x. Usually, an Ambrosetti–Prodi type result yields the existence of a number s_0 such that (1) has zero, at least one or at least two solutions according to $s < s_0$, $s = s_0$ or $s > s_0$. This terminology has become current after the founding work by A. Ambrosetti and G. Prodi [1] in 1972. Since then Ambrosetti–Prodi type results have been proved for several classes of boundary value problems: a thorough bibliography would include nearly two hundred titles.

In this contribution, based on the very recent paper [7], we analize the simplest case of the scalar periodic ODE

$$x' = f(t,x) \tag{2}$$

and the associated periodic Ambrosetti–Prodi problem

$$x' = f(t,x) - s. \tag{3}$$

Throughout we assume that $s \in \mathbb{R}$ is a parameter and

(h_1) *$f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is T-periodic with respect to the first variable and satisfies the L^1-Carathéodory conditions.*

Hereafter, by a T-periodic solution of (2) or (3) it is meant a T-periodic function $x : \mathbb{R} \to \mathbb{R}$ which is locally absolutely continuous and satisfies the equation for a.e. $t \in \mathbb{R}$.

Under the coercivity condition

$$f(t,x) \to +\infty, \ \text{ as } \ |x| \to +\infty \ \text{ uniformly a.e. in } t, \tag{4}$$

the periodic Ambrosetti–Prodi problem for (3) has been investigated by several authors, since the early eighties until very recent years: we refer to the bibliographies in [5, 6, 8] for a rather complete list of references. Thanks to its simplicity, (3) is in fact a quite good sample problem: manifold techniques can be effectively tested on it and the obtained results can suggest possible extensions to more general and complicated contexts.

In the case where f is a Bernoulli-type nonlinearity, i.e.,

(h_2) *there exist $a, b \in L^1(0,T)$ and $p > 0$ such that $f(t,x) = a(t)|x|^p + b(t)$ for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$,* the coercivity assumption (4) amounts to requiring that

$$\operatorname*{ess\,inf}_{[0,T]} a > 0.$$

However, when modeling, for instance, population dynamics, it is interesting to include cases where the function a vanishes on sets of positive measure or changes sign, in order to describe the occurrence of seasonal periods which inhibit or adversely affect the growth rate of the population under consideration. A real outbreak of papers devoted to the study of nonlinear problems which are indefinite in sign dates back to the eighties of the last century both in the PDEs and the ODEs settings, together with a parallel renewed interest towards ecological models (see, e.g., the monograph [2]).

First relevant progresses in relaxing the uniform coercivity assumption (4) were achieved in the recent papers [3, 8, 9]; precisely, the following result for equation (3) was obtained in [8].

Theorem 1 ([8, Theorem 3.3]). *Assume* (h_1),

(h_3) *there exist* $a, b \in L^1(0,T)$ *such that* $f(t,x) \geq a(t)|x| + b(t)$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}$,

(h_4) *there exists* $\overline{x} \in \mathbb{R}$ *such that* $\operatorname*{ess\,sup}_{t\in[0,T]} f(t,\overline{x}) < +\infty$,

(h_5) *for every* $K_1, K_2, \sigma \in \,]0,+\infty[$, *there exists* $d > 0$ *such that, for every* $x \in C^0([0,T])$ *with* $x(0) = x(T)$, *if*

$$\max_{[0,T]} |x| \leq K_1 \min_{[0,T]} |x| + K_2 \tag{5}$$

and either $\min\limits_{[0,T]} x \geq d$ *or* $\max\limits_{[0,T]} x \leq -d$, *then* $\int\limits_0^T f(t,x)\,dt > \sigma$.

Then, there exists $s_0 \in \mathbb{R}$ *such that equation* (3) *has zero, at least one or at least two* T*-periodic solutions according to* $s < s_0$, $s = s_0$ *or* $s > s_0$.

It is easy to check (see, e.g., [8, Corollary 4.1]) that (h_5) holds whenever the function a which appears in (h_3) satisfies both

(h_6) $\quad a(t) \geq 0$ *for a.e.* $t \in [0,T]$

and

(h_7) $\quad \int\limits_0^T a(t)\,dt > 0.$

Accordingly, condition (h_5) permits to consider nonlinearities which are just locally coercive, although bounded from below by a L^1-function.

In [7] we pushed further into the direction of relaxing the coercivity assumption on f, by showing that the non-negativity condition (h_6) can be dropped at all, while still achieving all the conclusions of Theorem 1. Namely, we proved the following result.

Theorem 2. *Assume* (h_1), (h_4),

(h_8) *there exist* $a, b \in L^1(0,T)$ *and* $p \in \,]0,1]$ *with* $f(t,x) \geq a(t)|x|^p + b(t)$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}$,

and (h_7). *Then, there exists* $s_0 \in \mathbb{R}$ *such that equation* (3) *has zero, at least one or at least two* T*-periodic solutions according to* $s < s_0$, $s = s_0$ *or* $s > s_0$.

Assumptions (h_8) and (h_7) basically require f being coercive on the average and allow that

$$\text{both} \quad \lim_{|x|\to+\infty} f(t,x) = +\infty \quad \text{and} \quad \lim_{|x|\to+\infty} f(t,x) = -\infty \ \text{ on sets of positive measure.}$$

It is worth stressing on the other hand that condition (h_5) prevents f from exhibiting this behavior, at least if f has the Bernoulli-type structure (h_2), as expressed by the following statement.

Proposition 3. *Assume* (h_2)*. Then, condition* (h_5) *is equivalent to conditions* (h_6) *and* (h_7)*.*

The proof of Theorem 2 is based on the direct construction of lower and upper solutions. Thus, from the results in [5], it is possible to infer various information on the qualitative properties of the obtained solutions. Indeed, for each $s > s_0$, equation (3) has at least one T-periodic solution which is weakly asymptotically stable from below, at least one T-periodic solution which is weakly asymptotically stable from above and at least one weakly stable T-periodic solution (all these solutions may possibly coincide), as well as, in addition, at least one unstable T-periodic solution, while for $s = s_0$ it has at least one unstable solution.

A question that may arise looking at Theorem (2) is whether or not one can assume $p > 1$ in condition (h_8). The answer is in general negative as shown by the following statement obtained in [7].

Proposition 4. *Assume* (h_1) *and*

(h_{10}) *there exist* $p > 1$, $I = [t_1, t_2] \subseteq [0, T]$ *and* $\delta > 0$ *such that* $f(t,x) \le -\delta|x|^p$ *for a.e.* $t \in I$ *and all* $x \in \mathbb{R}$*.*

Then, there exists $\sigma \in \mathbb{R}$ *such that, for all* $s \ge \sigma$*, equation* (3) *has no* T*-periodic solutions.*

In spite of the negative result of Proposition 4, we proved in [7] a positive result provided that $f(\cdot, 0) = 0$ and s is sufficiently small.

Proposition 5. *Assume* (h_1)*,*

(h_{11}) $f(\cdot, 0) = 0$ *and there exist* $a \in L^1(0,T)$ *and* $p > 1$ *such that* $f(t,x) \ge a(t)|x|^p$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}$*,*

and (h_7)*. Then, there exists* $\sigma > 0$ *such that, for all* $s \in\,]0, \sigma[$*, problem* (3) *has at least one positive* T*-periodic solution and at least one negative* T*-periodic solution.*

Open problem. It remains open the question of knowing if conclusions similar to the above can be proven for boundary value problems associated with second order ODEs or PDEs: a preliminary step in this direction is given by the perturbative result established in [3, Proposition 5.1].

References

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