Disconjugacy for the Fourth Order Ordinary Differential Equations

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In the paper, we study the question of the disconjugacy on the interval $I := [a, b] \subset [0, +\infty[$ of the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t), \tag{0.1}$$

where $p : I \to \mathbb{R}$ is Lebesgue integrable function.

Throughout the paper we use the following notations.

$\mathbb{R} =]-\infty, +\infty[\,$, $\mathbb{R}^+ =]0, +\infty[\,$, $\mathbb{R}_0^+ = [0, +\infty[\,$, $\mathbb{R}_0^- = \mathbb{R} \setminus \mathbb{R}^+$, $\mathbb{R}^- = \mathbb{R} \setminus \mathbb{R}_0^+$.

$C(I; \mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$.

$\widetilde{C}^3(I; \mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I; \mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

For arbitrary $x, y \in L(I; \mathbb{R})$, the notation

$$x(t) \preccurlyeq y(t) \ \ \big(x(t) \succcurlyeq y(t)\big) \ \text{ for } \ t \in I,$$

means that $x \le y$ $(x \ge y)$ and $x \ne y$. Also we use the notations $[x]_\pm = (|x| \pm x)/2$.

By a solution of equation (0.1) we understand a function $u \in \widetilde{C}^3(I; \mathbb{R})$, which satisfies equation (0.1) a.e. on I.

Definition 0.1. Equation (0.1) is said to be disconjugate (non oscillatory) on I, if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity. Otherwise, we say that equation (0.1) is oscillatory on I.

Definition 0.2. We say that $p \in D_+(I)$ if $p \in L(I; \mathbb{R}_0^+)$, and equation (0.1), under the boundary conditions

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0, 1), \tag{0.1_2}$$

has a solution u such that

$$u(t) > 0 \ \text{ for } \ t \in]a, b[\,. \tag{0.2}$$

Definition 0.3. We say that $p \in D_-(I)$ if $p \in L(I; \mathbb{R}_0^-)$, and equation (0.1) under the boundary conditions

$$u^{(i)}(a) = 0 \ \ (i = 0, 1, 2), \quad u(b) = 0 \tag{0.2_3}$$

has a solution u such that inequality (0.2) holds.

1 Main results

1.1 Disconjugacy of equation (0.1) with non-negative coefficient

First we consider equation (0.1) when the coefficient p is non-negative. In this case the following propositions are valid.

Theorem 1.1. *Let $p \in L(I;\mathbb{R}_0^+)$. Then equation (0.1) is disconjugate on I iff there exists $p^* \in D_+(I)$ such that*

$$p(t) \preccurlyeq p^*(t) \ \text{ for } \ t \in I. \tag{1.1}$$

Remark 1.1. From Theorem 1.1 it is clear that the structure of the set $D_+(I)$ is such that if $x, y \in D_+(I)$, then none of the inequalities $x \preccurlyeq y$ and $y \preccurlyeq x$ holds.

The following corollary shows us that for an arbitrary $p^* \in D_+(I)$, inequality (1.1) is optimal in some sense.

Corollary 1.1. *Let $p^* \in D_+(I)$ and*

$$p(t) \geq p^*(t) \ \text{ for } \ t \in I. \tag{1.2}$$

Then equation (0.1) is oscillatory on I.

Let $\lambda_1 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = \lambda^4 u(t), \quad u^{(i)}(0) = 0, \quad u^{(i)}(1) = 0 \ \ (i = 0, 1). \tag{1.3}$$

Then from Theorem 1.1 and Corollary 1.1 we obtain

Corollary 1.2. *Equation (0.1) is disconjugate on I if*

$$0 \leq p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \ \text{ for } \ t \in I,$$

and is oscillatory on I if

$$p(t) \geq \frac{\lambda_1^4}{(b-a)^4} \ \text{ for } \ t \in I.$$

Remark 1.2. It is well known that the first eigenvalue λ_1 of problem (1.3), is the first positive root of the equation $\cos\lambda \cdot \cosh\lambda = 1$, and $\lambda_1 \approx 4.73004$ (see [2,5]). Also, in Theorem 3.1 of paper [5] it was proved that the equation $u^{(4)} = \lambda^4 u$ is disconjugate on $[0,1]$ if $0 \leq \lambda < \lambda_1$.

1.2 Disconjugacy of equation (0.1) with non-positive coefficient

Now we consider equation (0.1) with the non-positive coefficient p, for which the following propositions are valid.

Theorem 1.2. *Let $p \in L(I;\mathbb{R}_0^-)$. Then equation (0.1) is disconjugate on I iff there exists $p_* \in D_-(I)$ such that*

$$p(t) \succcurlyeq p_*(t) \ \text{ for } \ t \in I. \tag{1.4}$$

Remark 1.3. From Theorem 1.2 it is clear that the structure of the set $D_-(I)$ is such that if $x, y \in D_-(I)$, then none of the inequalities $x \preccurlyeq y$ and $y \preccurlyeq x$ holds.

The following corollary shows us that for an arbitrary $p^* \in D_-(I)$, inequality (1.4) is optimal in some sense.

Corollary 1.3. *Let $p_* \in D_-(I)$, and*

$$p(t) \le p_*(t) \;\; \text{for} \;\; t \in I. \tag{1.5}$$

Then equation (0.1) *is oscillatory on I.*

Let $\lambda_2 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = -\lambda^4 u(t), \quad u^{(i)}(0) = 0 \;\; (i = 0, 1, 2), \quad u(1) = 0. \tag{1.6}$$

Then from Theorem 1.2 and Corollary 1.3 we obtain

Corollary 1.4. *Equation* (0.1) *is disconjugate on I if*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \le 0 \;\; \text{for} \;\; t \in I,$$

and is oscillatory on I if

$$p(t) \le -\frac{\lambda_2^4}{(b-a)^4} \;\; \text{for} \;\; t \in I.$$

Remark 1.4. In Theorem 4.1 of paper [5] (see also [2, Theorems 3.5 and 3.6], [1, Subsection 4.1]) following is proved: let λ_2 be the first positive root of the equation $\tanh \frac{\lambda}{\sqrt{2}} = \tan \frac{\lambda}{\sqrt{2}}$ ($\lambda_2 \approx 5.553$). Then the equation $u^{(4)} = -\lambda^4 u$ is disconjugate on $[0,1]$ if $0 \le \lambda < \lambda_2$.

1.3 Disconjugacy of equation (0.1) with not necessarily constant sign coefficient

On the basis of Theorems 1.1 and 1.2, we can get the non-improvable results which guarantee the diconjugacy of equation (0.1) on I, when p is not necessarily constant sign function.

Theorem 1.3. *Let $p_* \in D_-(I)$ and $p^* \in D_+(I)$. Then for an arbitrary function $p \in L(I;\mathbb{R})$, such that*

$$p_*(t) \preccurlyeq -[p(t)]_-, \quad [p(t)]_+ \preccurlyeq p^*(t) \;\; \text{for} \;\; t \in I, \tag{1.7}$$

equation (0.1) *is disconjugate on I.*

The theorem is optimal in the sense that inequalities (1.7) can not be replaced by the condition $p_* \le p \le p^*$.

Corollary 1.5. *Let the functions $p_1 \in L(I;\mathbb{R}_0^-)$, $p_2 \in L(I;\mathbb{R}_0^+)$, be such that the equations*

$$u^{(4)}(t) = p_1(t)u(t), \quad u^{(4)}(t) = p_2(t)u(t) \tag{1.8}$$

are disconjugate on I, and

$$p_1(t) \le p(t) \le p_2(t) \;\; \text{for} \;\; t \in I. \tag{1.9}$$

Then equation (0.1) *is disconjugate on I.*

Remark 1.5. We can see that in Kondrat'ev's comparison second theorem:

Theorem 1.4 ([4, Theorem 2]). *Let the continuous functions $p_1, p_2 : [a,b] \to \mathbb{R}$ be such that equations* (1.8) *are disconjugate on I, and $p_1 \le p \le p_2$. Then equation* (0.1) *is disconjugate too.*

The permissible coefficients p_1 and p_2 should not necessarily be constant sign functions, while in Theorem 1.3 for the permissible coefficients p_1 and p_2 equations (1.8) should not necessarily be disconjugate. For this reason, for example, if $p(t) = \lambda_1^4[\cos(2\pi t/n)]_+ - \lambda_2^4[\cos(2\pi t/n)]_-$, then from Theorem 1.3 it follows the disconjugacy of equation (0.1) on $[0,1]$ for all $n \in N$ (see Corollary 1.6), while this fact does not follows from Kondrat'ev's theorem.

From Theorem 1.3 with $p_* := -\frac{\lambda_2^4}{(b-a)^4}$ and $p^* := \frac{\lambda_1^4}{(b-a)^4}$ we obtain

Corollary 1.6. *Let $\lambda_1 > 0$ and $\lambda_2 > 0$ be the first eigenvalues of problems* (1.3) *and* (1.6), *respectively, and the function $p \in L(I;\mathbb{R})$ admits the inequalities*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \quad \textit{for } t \in I.$$

Then equation (0.1) *is disconjugate on I.*

Remark 1.6. If we take into account that $\lambda_1^4 \approx 501$ and $\lambda_2^4 \approx 951$, then it is clear that Corollary 1.6 significantly improves W. Coppel's well-known condition

$$\max_{t\in[a,b]} |p(t)| \leq \frac{128}{(b-a)^4}\,,$$

proved in [3, Theorem 1, p. 86], which for $p \in C(I;\mathbb{R})$ guarantees the disconjugacy of equation (0.1) on I.

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