How to Assign Lyapunov Spectra for Completely Controllable Systems

E. K. Makarov

Institute of Mathematics of the National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: jcm@im.bas-net.by

S. N. Popova

Udmurt State University, Izhevsk, Russia
E-mail: udsu.popova.sn@gmail.com

Consider a linear controllable differential system

$$\dot{x} = A(t)x + B(t)u, \ \ x \in \mathbb{R}^n, \ \ u \in \mathbb{R}^m, \ \ t \geq 0, \tag{1}$$

with piecewise continuous and bounded coefficient matrices A and B. We denote the Cauchy matrix of the corresponding free system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

by $X(t,s)$, where $t,s \geq 0$, and the Lyapunov exponents of (2) by $\lambda_k(A)$, $k = 1,\dots,n$.

Suppose that the control u is formed as a linear feedback $u = U(t)x$, where the matrix U is also piecewise continuous and bounded. Then the closed-loop system

$$\dot{y} = \big(A(t) + B(t)U(t)\big)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{3}$$

should be treated as a linear differential system with bounded piecewise continuous coefficients. So, all Lyapunov invariants (i.e. invariants of Lyapunov transformations) including Lyapunov exponents $\lambda_k(A+BU)$, $k = 1,\dots,n$, are defined for system (3). Further we assume that the Lyapunov exponents of each system (both (1) and (3)) are arrayed in increasing order as follows

$$\lambda_1(A) \leq \cdots \leq \lambda_n(A)$$

and, respectively,

$$\lambda_1(A+BU) \leq \cdots \leq \lambda_n(A+BU).$$

According to classical definition due to Kalman [1] system (1) is said to be uniformly completely controllable if there exist positive real numbers ϑ and α_i, $i = 1,\dots,4$, such that for all $\tau \in \mathbb{R}$ the inequalities

$$\alpha_1 I \leqslant W(\tau,\tau+\vartheta) \leqslant \alpha_2 I, \tag{4}$$

$$\alpha_3 I \leqslant \widehat{W}(\tau,\tau+\vartheta) \leqslant \alpha_4 I \tag{5}$$

hold. Here the controllability matrix W (Kalman matrix) is given by

$$W(t_0,t_1) = \int\limits_{t_0}^{t_1} X(t_0,s)B(s)B^T(s)X^T(t_0,s)\,ds,$$

$\widehat{W}(t_0,t_1) = X(t_1,t_0)W(t_0,t_1)X^T(t_1,t_0)$, and $I \in \mathbb{R}^{n\times n}$ is the identity matrix.

System (1) is said to be ϑ-uniformly completely controllable for some given $\vartheta > 0$ if the conditions of the above definition are satisfied for this value of ϑ.

The matrix inequalities (4) and (5) should be understood as a conditional notation of inequalities between corresponding quadratic forms. Namely, conditions (4) and (5) mean that the inequalities

$$\alpha_1\|h\|^2 \leqslant h^T W(\tau,\tau+\vartheta)h = \int\limits_{\tau}^{\tau+\vartheta} \big\|h^T X(\tau,s)B(s)\big\|^2\,ds \leqslant \alpha_2\|h\|^2,$$

$$\alpha_3\|h\|^2 \leqslant h^T \widehat{W}(\tau,\tau+\vartheta)h = \int\limits_{\tau}^{\tau+\vartheta} \big\|h^T X(\tau+\vartheta,s)B(s)\big\|^2\,ds \leqslant \alpha_4\|h\|^2$$

are valid for all $h \in \mathbb{R}^n$.

Since coefficients of system (1) are piecewise continuous and bounded, we can equivalently reformulate Kalman's definition in a somewhat simpler way as follows. System (1) is uniformly completely controllable if there exist positive real numbers ϑ and α such that for all $\tau \in \mathbb{R}$ the inequalities

$$W(\tau,\tau+\vartheta) \geqslant \alpha I$$

hold. In this case an alternative form for definition of uniform complete controllability was given by E. L. Tonkov in [3]. We say that system (1) is ϑ-uniformly completely controllable if there exists a number $l > 0$ such that for any state x_0 and each segment $[\tau,\tau+\vartheta]$ there exist a control u ensuring the transfer of system (1) from x_0 to 0 on this segment and satisfying the condition $\|u(t)\| \leqslant l\|x_0\|$ for all $t \in [\tau,\tau+\vartheta]$. For systems with piecewise continuous and bounded coefficients both of the definitions are equivalent.

The following result is well known in the theory of control of asymptotic invariants [2, p. 337].

Theorem 1. *If system* (1) *is uniformly completely controllable and the matrix B is piecewise uniformly continuous (i.e. B can be represented as a sum of uniformly continuous and piecewise constant matrices), then the Lyapunov exponents of system* (3) *are globally controllable.*

Recall that the Lyapunov exponents of system (3) are said to be globally controllable if for any given $\mu_k \in \mathbb{R}$, $k = 1,\dots,n$, such that $\mu_1 \le \cdots \le \mu_n$ there exists a bounded piecewise continuous feedback matrix U such that the equalities $\lambda_k(A+BU) = \mu_k$ are valid for all $k = 1,\dots,n$. These facts motivate us to refer to the matrix U as a matrix control.

If system (1) is completely controllable, but is not uniformly completely controllable, then for any initial time $t_0 \ge 0$ there exists a $t_1(t_0) > t_0$ such that for any state x_0 of system (1) one can find a control u steering the system from x_0 to zero state on the interval $[t_0, t_1(t_0)]$. Note that in this case no condition is posed on the norm of the control function u and the length of the segment $[t_0, t_1(t_0)]$ is allowed to grow indefinitely when the starting point t_0 moves away from zero.

Proving Theorem 1 we evaluate the Lyapunov exponents of system (3) along the sequence $k\vartheta$, $k \in \mathbb{N}$, and ensure boundedness of matrix control using the property provided by definition of uniform complete controllability in the form due to Tonkov. So we have to conclude that approaches used to prove Theorem 1 are not suitable to solve the Lyapunov spectra assignment problem when the original system (1) is not uniformly completely controllable.

The most natural way to overcome this problem is to use more rapidly growing sequences. In this case, as before, we retain the ability to construct the required control by the aid of the matrix system

$$\dot{X} = A(t)X + B(t)V, \ \ X \in \mathbb{R}^{n\times n}, \ \ V \in \mathbb{R}^{m\times n}, \ \ t \ge 0,$$

corresponding to system (1).

To implement this idea we introduce two functions describing some controllability properties of system (1). For each $t \in \mathbb{R}$, $t > 0$, by $T(t)$ we denote the exact lower bound of the set of all $\tau \in \mathbb{R}$, $\tau > t$, such that system (1) is completely controllable on $[t, \tau]$.

For each $t, s \in \mathbb{R}$, where $t > T(s) \geqslant s$, by $\Gamma(t, s)$ we denote the exact lower bound of the set of all numbers γ such that for any state x_0 of system (1) there exists a control u steering the system from x_0 to 0 on the segment $[s, t]$ and satisfying the estimate $\|u(\tau)\| \leqslant \gamma\|x_0\|$ for all $\tau \in [s, t]$. Additionally, we assume $\Gamma(t, s) = +\infty$ when $t \leqslant T(s)$.

We say that the sequence t_k, $k \in \mathbb{N}$, satisfies the slow growth condition if $\lim\limits_{k\to\infty} \frac{t_{k+1}}{t_k} = 1$.

Example. Consider a scalar system

$$\dot{x} = b(t)u, \quad x \in \mathbb{R}, \quad u \in \mathbb{R}, \quad t \geq 0, \tag{6}$$

having the form (1) with zero 1×1-matrix A. Let $t_k > 0$, $k \in \mathbb{N}$ be a monotonically increasing to $+\infty$ sequence and s_k, $k \in \mathbb{N}$ be a sequence satisfying the condition $t_{k-1} < s_k < t_k$ for all $k \geqslant 2$.

Let us define a scalar function b as follows: $b(t) = 1$, $t \in]s_k, t_k]$, and $b(t) = 0$ for all other $t \geq 0$. It can be easily proved that system (6) is completely controllable and is not uniformly completely controllable if the sequence $s_k - t_{k-1}$ is unbounded. By direct calculation we assert that $T(t) = s_k$ for $t \in [t_{k-1}, s_k]$ and $T(t) = t$ for $t \in [s_k, t_k]$. Moreover, for $s \in [t_{k-1}, s_k]$, $t \in]s_k, t_k]$ we have $\Gamma(t, s) = (t - s_k)^{-1}$.

The following statements are valid.

(i) If $k/t_k \to 0$ as $k \to \infty$ and the sequence $t_k - s_k$ is bounded, then the Lyapunov exponent of system (6) equals to zero whatever control we choose.

(ii) If $k/t_k \to 0$ as $k \to \infty$ and the sequence s_k satisfies the condition $s_k = \mu t_{k-1} + (1 - \mu)t_k$ with some $\mu \in]0, 1[$, then choosing an appropriate control u we can prescribe any value for the exponent of system (6). Note that we can choose u to be a constant.

Theorem 2. *Suppose that system* (1) *is completely controllable and the matrix B is piecewise uniformly continuous. If there exists a monotonically increasing to $+\infty$ sequence t_k, $k \in \mathbb{N}$, of positive real numbers satisfying the slow growth condition and such that for some $\alpha > 0$ the inequalities*

$$\Gamma(t_{k+1}, t_k) \leqslant \alpha(t_{k+1} - t_k)^{-1},$$

are valid for all $k \in \mathbb{N}$, then the Lyapunov exponents of system (3) *are globally controllable.*

Remark. If the sequence t_k does not satisfy the slow growth condition, then our ability to assign the Lyapunov spectrum of system (3) depends on finer asymptotic properties of free system (2).

Corollary 1. *Suppose that system* (1) *is completely controllable and the matrix B is piecewise uniformly continuous. If there exists a monotonically increasing to $+\infty$ sequence t_k, $k \in \mathbb{N}$, of positive real numbers satisfying the slow growth condition such that for some $\gamma > 0$ the inequalities*

$$W(t_k, t_{k+1}) \geqslant \gamma(t_{k+1} - t_k)I,$$

are valid for all $k \in \mathbb{N}$, then the Lyapunov exponents of system (3) *are globally controllable.*

To prove Corollary 1 we use the standard Kalman controls [1] existing on each segment where some controllable system is completely controllable. These control functions are useless for immediate constructing of necessary matrix controls, but their norms satisfy an appropriate estimate to apply Theorem 2.

References

[1] R. E. Kalman, Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana (2)* **5** (1960), 102–119.

[2] E. K. Makarov and S. N. Popova, *Controllability of Asymptotic Invariants of Non-Stationary Linear Systems.* (Russian) Navuka, Minsk, Belarus, 2012.

[3] E. L. Tonkov, A criterion of uniform controllability and stabilization of a linear recurrent system. (Russian) *Differentsial'nye Uravneniya* **15** (1979), no. 10, 1804–1813.