On Positive Periodic Solutions to Parameter-Dependent Second-Order Differential Equations with a Singularity

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We are interested in the existence and non-existence of a **positive** solution to the periodic boundary value problem

$$\boxed{u'' = p(t)u - \frac{h(t)}{u^{\lambda}} + \mu f(t); \quad u(0) = u(\omega), \quad u'(0) = u'(\omega).} \tag{0.1}$$

Here, $p, h, f \in L([0, \omega])$,

$$h(t) \geq 0 \ \text{ for a.e. } \ t \in [0, \omega], \quad h(t) \not\equiv 0,$$

$\lambda > 0$, and a parameter $\mu \in \mathbb{R}$. By a *solution* to problem (0.1), as usual, we understand a function $u\colon [0, \omega] \to]0, \infty[$ which is absolutely continuous together with its first derivative, satisfies the given equation almost everywhere, and meets periodic conditions.

Definition 0.1. We say that the function $p \in L([0, \omega])$ belongs to the set $\mathcal{V}^{+}(\omega)$ (resp. $\mathcal{V}^{-}(\omega)$) if for any function $u \in AC^1([0, \omega])$ satisfying

$$u''(t) \geq p(t)u(t) \ \text{ for a.e. } \ t \in [0, \omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \ \text{ for } \ t \in [0, \omega] \quad \big(\text{resp. } \ u(t) \leq 0 \ \text{ for } \ t \in [0, \omega]\big)$$

is fulfilled.

Definition 0.2. We say that the function $p \in L([0, \omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{0.2}$$

has a positive solution.

For the cases $p \in \mathcal{V}^{-}(\omega)$, $p \in \mathcal{V}_0(\omega)$, and $p \in \mathcal{V}^{+}(\omega)$, we provide some results concerning the existence or non-existence of solutions to problem (0.1) depending on the choice of a parameter μ.

1 The case $p \in \mathcal{V}^-(\omega)$

Theorem 1.1. *Let $p \in \mathcal{V}^-(\omega)$. Then, there exist $-\infty \le \mu_* < 0$ and $0 < \mu^* \le +\infty$ such that*

- *for any $\mu \in \,]\mu_*, \mu^*[\,$, problem (0.1) has a unique solution,*
- *if $\mu_* > -\infty$, then, for any $\mu \le \mu_*$, problem (0.1) has no solution,*
- *if $\mu^* < +\infty$, then, for any $\mu \ge \mu^*$, problem (0.1) has no solution.*

2 The case $p \in \mathcal{V}_0(\omega)$

Theorem 2.1. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\int\limits_0^\omega f(t)u_0(t)\,\mathrm{d}t > 0,$$

where u_0 is a solution to problem (0.2). Then, there exists $0 < \mu^* \le +\infty$ such that

- *for any $\mu \le 0$, problem (0.1) has no positive solution,*
- *for any $\mu \in \,]0, \mu^*[\,$, problem (0.1) has a unique solution,*
- *if $\mu^* < +\infty$, then, for any $\mu \ge \mu^*$, problem (0.1) has no solution.*

From Theorem 2.1, we derive immediately the following result.

Theorem 2.2. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\int\limits_0^\omega f(t)u_0(t)\,\mathrm{d}t < 0,$$

where u_0 is a solution to problem (0.2). Then, there exists $-\infty \le \mu_* < 0$ such that

- *if $\mu_* > -\infty$, then, for any $\mu \le \mu_*$, problem (0.1) has no solution,*
- *for any $\mu \in \,]\mu_*, 0[\,$, problem (0.1) has a unique solution,*
- *for any $\mu \ge 0$, problem (0.1) has no positive solution.*

3 The case $p \in \mathcal{V}^+(\omega)$

Remark 3.1. In [1, Theorem 16.4], it is shown that, if $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and

$$\int\limits_0^\omega [f(t)]_+\,\mathrm{d}t > \nu^*(p)\bigg(\frac{\omega}{4}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s\bigg)\int\limits_0^\omega [f(t)]_-\,\mathrm{d}t, \tag{3.1}$$

where the number $\nu^*(p)$ depends only on p (see [1, formula (6.22)]), then the linear periodic problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

possesses a unique solution u which is positive.

Theorem 3.1. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and (3.1) hold. Then, there exists $0 \le \mu_* < \infty$ such that*

- *for any* $\mu > \mu_*$, *problem* (0.1) *has a solution,*
- *if* $\mu_* > 0$, *then, for any* $\mu < \mu_*$, *problem* (0.1) *has no solution,*
- *if* $\mu_* = 0$, *then, for any* $\mu \le 0$, *problem* (0.1) *has no solution.*

From Theorem 3.1, we derive immediately the following result.

Theorem 3.2. *Let* $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ *and*

$$\int_0^\omega [f(t)]_- \,\mathrm{d}t > \nu^*(p)\bigg(\frac{\omega}{4}\int_0^\omega [p(s)]_+ \,\mathrm{d}s\bigg)\int_0^\omega [f(t)]_+ \,\mathrm{d}t,$$

Then, there exists $-\infty < \mu^* \le 0$ *such that*

- *for any* $\mu < \mu^*$, *problem* (0.1) *has a solution,*
- *if* $\mu^* < 0$, *then, for any* $\mu > \mu^*$, *problem* (0.1) *has no solution,*
- *if* $\mu^* = 0$, *then, for any* $\mu \ge 0$, *problem* (0.1) *has no solution.*

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References

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