On the Existence of Some Solutions of Systems of Ordinary Differential Equations that are Partially Resolved Relatively to the Derivatives with Square Matrix

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Let us consider the system of ordinary differential equations

$$A(z)Y' = B(z)Y + f(z,Y,Y'), \tag{1}$$

where the matrices $A : D_1 \to \mathbb{C}^{p\times p}$, $B : D_{10} \to \mathbb{C}^{p\times p}$, $D_1 = \{z : \ |z| < R_1, \ R_1 > 0\} \subset \mathbb{C}$, $D_{10} = D_1 \setminus \{0\}$, matrix $A = A(z)$ is analytical in the domain D_1, matrix $B = B(z)$ is analytical in the domain D_{10}, $\operatorname{rang} A(z) = p$ in the domain $z \in D_1$, $A^{(-1)}(z)B(z)$ is analytical matrix in the domain D_{10} and has pole of order $d \in \mathbb{N}$ in the point $z = 0$, the vector-function $f : D_1\times G_1\times G_2 \to \mathbb{C}^p$, where domains $G_k \subset \mathbb{C}^p$, $0 \in G_k$, $k = 1, 2$, the vector-function $f = f(z,Y,Y')$ is analytical in the domain $D_{10} \times G_{10} \times G_{20}$, $G_{k0} = G_k \setminus \{0\}$, $k = 1, 2$, the decomposition of the vector function $f = f(z,Y,Y')$ to a convergent power series around the point $(0, 0, 0)$ has no free and linear members.

Let us study question on the existence of analytic solutions of the Cauchy problem for system (1) with the initial condition

$$Y \to 0, \ \ z \to 0, \ \ z \in D_{10},$$

and the additional condition

$$Y' \to 0, \ \ z \to 0, \ \ z \in D_{10}.$$

According to these assumptions, system (1) takes the form

$$z^dY' = \check{P}^{(2)}(z)Y + z^dH^{(2)}(z,Y,Y'), \tag{2}$$

where $\check{P}^{(2)}(z)$ is an analytical matrix in the domain D_1, $H^{(2)} = H^{(2)}(z,Y,Y')$ is an analytical vector-function in the domain $D_1 \times G_1 \times G_2$.

Definition 1. Let's define that the vector-function $z^dH^{(2)}(z,Y,Y')$ has the property V_1 near the point $(0, 0, 0)$ if this neighborhood component vector function $z^dH^{(2)}(z,Y,Y')$ may be decomposed into convergent series form

$$z^dH^{(2)}_j(z,Y,Y') = \sum_{s+|l|+|q|=2}^{\infty} C^{(2.j)}_{slq} z^sY^l(z^dY')^q, \ \ j = \overline{1,p},$$

where $C^{(2.j)}_{slq} \in \mathbb{C}$, $j = \overline{1,p}$.

Lemma. *If in system* (2) *vector-function* $z^dH^{(2)}(z,Y,Y')$ *has the property* V_1 *near the point* $(0, 0, 0)$*, then system* (2) *can be uniquely reduced to the system of the type*

$$z^dY' = P^{(2)}(z)Y + F^{(2)}(z,Y), \tag{3}$$

where $P^{(2)}(z)$ *is an analytical matrix in the domain* $\widetilde{D_1} \subseteq D_1$, $0 \in \widetilde{D_1}$, $F^{(2)} = F^{(2)}(z,Y)$ *is an analytical vector-function in the domain* $\widetilde{D_1} \times \widetilde{G_1} \subseteq D_1 \times G_1$, $(0, 0) \in \widetilde{D_1} \times \widetilde{G_1}$, $F^{(2)}(0, 0) = 0$. *For convenience, we assume that the matrix* $P^{(2)}$ *is analytical in the domain* D_1*, and the vector-function* $F^{(2)}$ *is analytical in the domain* $D_1 \times G_1$.

For arbitrarily fixed $t_1 \in (0, R_1]$, $v_1, v_2 \in \mathbb{R}$, $v_1 < v_2$, introduce a set $\check{I}(t_1) = \{(t, v) \in \mathbb{R}^2 :\ t \in (0, t_1),\ v \in (v_1, v_2)\}$. For $z = z(t, v) = te^{iv}$, the set $\check{I}(t_1) \subset \mathbb{R}^2$ refers to the set $I(t_1) \subset \mathbb{C} :\ I(t_1) = \{z = te^{iv} \in \mathbb{C} :\ t \in (0, t_1),\ v \in (v_1, v_2)\}$.

Definition 2. Let $p, g : \check{I}(t_1) \to [0, +\infty)$. Let's define that the function p has the property Q_1 regarding the function g on the condition $v = v_0 \in (v_1, v_2)$, if the function $p = p(t, v_0)$ is a function of higher order of smallness relative to the function $g = g(t, v_0)$ on the condition $t \to +0$.

Definition 3. Let $p, g : \check{I}(t_1) \to [0, +\infty)$. Let's define that the function p has the property Q_2 regarding the function g on the set $\check{I}(t_1)$, if there exist $C_1 \geq 0$, $C_2 \geq 0$ such that on the set $\check{I}(t_1)$ the inequality

$$C_1 g(t, v) \leq p(t, v) \leq C_2 g(t, v)$$

is satisfied.

Introduce the auxiliary vector function $\varphi(z) = \operatorname{col}(\varphi_1(z), \dots, \varphi_p(z))$, $\varphi : I(t_1) \to \mathbb{C}^p$, and $\psi(t, v) = \operatorname{col}(\psi_1(t, v), \dots, \psi_p(t, v))$, $\psi_j : \check{I}(t_1) \to [0; +\infty)$, $j = \overline{1, p}$, on the condition $z = z(t, v) = te^{iv}$, $\psi_j(t, v) = |\varphi_j(z(t, v))|$, $j = \overline{1, p}$, functions ψ_j, $j = \overline{1, p}$ are really values functions of real variables t, v.

For a fixed $v = v_0$ we introduce

$$\begin{gathered}
Y(z(t, v_0)) = \widetilde{Y}(t), \quad \widetilde{Y}(t) = \widetilde{Y}_1(t) + i\widetilde{Y}_2(t), \\
P^{(2)}(z(t, v_0)) = \|\widetilde{p}^{(2)}_{jk}(t)\|^p_{j,k=1} = \widetilde{P}^{(2)}_1(t) + i\widetilde{P}^{(2)}_2(t), \quad \widetilde{P}^{(2)}_s(t) = \|\widetilde{p}^{(2)}_{jks}(t)\|^p_{j,k=1}, \ \ s = 1, 2, \\
F^{(2)}\big(z(t, v_0), Y(z(t, v_0))\big) = \widetilde{F}^{(2)}(t, \widetilde{Y}_1, \widetilde{Y}_2), \\
\widetilde{F}^{(2)}(t, \widetilde{Y}_1, \widetilde{Y}_2) = \operatorname{col}\big(\widetilde{F}^{(2)}_1(t, \widetilde{Y}_1, \widetilde{Y}_2), \dots, \widetilde{F}^{(2)}_p(t, \widetilde{Y}_1, \widetilde{Y}_2)\big), \\
\widetilde{F}^{(2)}_j(t, \widetilde{Y}_1, \widetilde{Y}_2) = \widetilde{F}^{(2)}_{1j}(t, \widetilde{Y}_1, \widetilde{Y}_2) + i\widetilde{F}^{(2)}_{2j}(t, \widetilde{Y}_1, \widetilde{Y}_2), \ \ j = \overline{1, p},
\end{gathered}$$

functions $\widetilde{p}^{(2)}_{jks}(t)$, $j, k = \overline{1, p}$, $s = 1, 2$, and vector-functions $\widetilde{Y}_1(t)$, $\widetilde{Y}_2(t)$, $\widetilde{F}^{(2)}_{1j}$, $\widetilde{F}^{(2)}_{2j}$, $j = \overline{1, p}$ are really values functions of real variable t.

For a fixed $t = t_0$ we introduce

$$\begin{gathered}
Y(z(t_0, v)) = \widehat{Y}(v) = \widehat{Y}_1(v) + i\widehat{Y}_2(v), \\
P^{(2)}(z(t_0, v)) = \|\widehat{p}^{(2)}_{jk}(v)\|^p_{j,k=1} = \widehat{P}^{(2)}_1(v) + i\widehat{P}^{(2)}_2(v), \quad \widehat{P}^{(2)}_s(v) = \|\widehat{p}^{(2)}_{jks}(v)\|^p_{j,k=1}, \ \ s = 1, 2, \\
F^{(2)}\big(z(t_0, v), Y(z(t_0, v))\big) = \widehat{F}^{(2)}(v, \widehat{Y}_1, \widehat{Y}_2), \\
\widehat{F}^{(2)}(v, \widehat{Y}_1, \widehat{Y}_2) = \operatorname{col}\big(\widehat{F}^{(2)}_1(v, \widehat{Y}_1, \widehat{Y}_2), \dots, \widehat{F}^{(2)}_p(v, \widehat{Y}_1, \widehat{Y}_2)\big), \\
\widehat{F}^{(2)}_j(v, \widehat{Y}_1, \widehat{Y}_2) = \widehat{F}^{(2)}_{1j}(v, \widehat{Y}_1, \widehat{Y}_2) + i\widehat{F}^{(2)}_{2j}(v, \widehat{Y}_1, \widehat{Y}_2), \ \ j = \overline{1, p},
\end{gathered}$$

functions $\widehat{p}^{(2)}_{jks}(v)$, $j, k = \overline{1, p}$, $s = 1, 2$, and vector-functions $\widehat{Y}_1$, $\widehat{Y}_2$, $\widehat{F}^{(2)}_{1j}$, $\widehat{F}^{(2)}_{2j}$, $j = \overline{1, p}$ are really values functions of real variable v.

Definition 4. Let's define that the matrix $P^{(2)}(z)$ has the property S_2 regarding the vector-function $\varphi = \varphi(z)$ if the conditions are met:

1) for each $v_0 \in (v_1, v_2)$ functions $t^d(\psi_j(z(t, v)))'_t$ have the property Q_1 regarding the functions $|\widetilde{p}^{(2)}_{jj}(t)|\psi_j(z(t, v))$, $j = \overline{1, p}$, on the condition $v = v_0$;

2) functions $t^{d-1}(\psi_j(t, v))'_v$ have the property Q_2 regarding the functions $|\widehat{p}^{(2)}_{jj}(v)|\psi_j(t, v)$, $j = \overline{1, p}$, on the set $\check{I}(t_2)$ for some $t_2 \in (0, t_1)$;

3) for each $v_0 \in (v_1, v_2)$ functions $|\widetilde{p}^{(2)}_{jk}(t)|\psi_k(t,v)$ have the property Q_1 regarding the functions $t^d(\psi_j(t,v))'_t$, $j,k = \overline{1,p}$, $j \neq k$, on the condition $v = v_0$;

4) functions $|\widehat{p}^{(2)}_{jk}(v)|\psi_k(t,v)$ have the property Q_2 regarding the functions $t^{d-1}(\psi_j(t,v))'_v$, $j,k = \overline{1,p}$, $j \neq k$, on the set $\check{I}(t_2)$ for some $t_2 \in (0,t_1)$.

Let's introduce the sets

$$\widetilde{\Omega}\big(\delta, \varphi(z(t,v_0))\big) = \Big\{(t, \widetilde{Y}_1, \widetilde{Y}_2): \ t \in (0,t_1), \ \widetilde{Y}^2_{1j} + \widetilde{Y}^2_{2j} < \delta^2_j(\psi_j(t,v_0))^2, \ j = \overline{1,p}\Big\},$$

v_0 is fixed on the interval (v_1, v_2),

$$\widehat{\Omega}\big(\tau, \varphi(z(t_0,v))\big) = \Big\{(v, \widehat{Y}_1, \widehat{Y}_2): \ v \in (v_1,v_2), \ \widehat{Y}^2_{1j} + \widehat{Y}^2_{2j} < \tau^2_j(\psi_j(t_0,v))^2, \ j = \overline{1,p}\Big\},$$

t_0 is fixed on the interval $(0,t_1)$, where $\delta = (\delta_1, \dots, \delta_p)$, $\tau = (\tau_1, \dots, \tau_p)$, $\delta_j, \tau_j \in \mathbb{R} \setminus \{0\}$, $j = (1,p)$.

Definition 5. Let's define that the vector-function $F^{(2)} = F^{(2)}(z,Y)$ has the property M_2 regarding the vector-function $\varphi = \varphi(z)$ if the conditions are met:

1) for each $v_0 \in (v_1,v_2)$ on the condition $(t, \widetilde{Y}_1, \widetilde{Y}_2) \in \widetilde{\Omega}(\delta, \varphi(z(t,v_0)))$ functions $\widetilde{F}^{(2)}_{kj} = \widetilde{F}^{(2)}_{kj}(t, \widetilde{Y}_1, \widetilde{Y}_2)$ have the property Q_1 regarding the functions $|\widetilde{p}^{(2)}_{jj}(t)|\psi_j(t,v)$, $j = \overline{1,p}$, $k = 1,2$, on the condition $v = v_0$;

2) for each $(v, \widehat{Y}_1, \widehat{Y}_2) \in \widehat{\Omega}(\tau, \varphi(z(t_0,v)))$ functions $\widehat{F}^{(2)}_{kj} = \widehat{F}^{(2)}_{kj}(v, \widehat{Y}_1, \widehat{Y}_2)$ have the property Q_2 regarding the function $|\widehat{p}^{(2)}_{jj}(v)|\psi_j(t,v))$, $j = \overline{1,p}$, $k = 1,2$, on the set $\check{I}(t_2)$ for some $t_2 \in (0,t_1)$.

Let's introduce domains $\Lambda^{(2)}_{+.k}(t_2)$, $k \in \{+,-\}$, which are defined as

$$\Lambda^{(2)}_{+.+}(t_2) = \Big\{(t,v) : \cos\big((d-1)v - \widetilde{\alpha}^{(2)}_{jj}(t)\big) > 0, \ \sin\big((d-1)v - \widehat{\alpha}^{(2)}_{jj}(v)\big) > 0, \\ j = \overline{1,p}, \ t \in (0,t_2), \ v \in (v_1,v_2)\Big\},$$

$$\Lambda^{(2)}_{+.-}(t_2) = \Big\{(t,v) : \ \cos\big((d-1)v - \widetilde{\alpha}^{(2)}_{jj}(t)\big) > 0, \ \sin\big((d-1)v - \widehat{\alpha}^{(2)}_{jj}(v)\big) < 0, \\ j = \overline{1,p}, \ t \in (0,t_2), \ v \in (v_1,v_2)\Big\},$$

where functions $\widetilde{\alpha}^{(2)}_{jj}(t)$, $\widehat{\alpha}^{(2)}_{jj}(v)$, $j = \overline{1,p}$, are defined through the corresponding diagonal elements of the matrices $\widetilde{P}^{(2)}_q$, $\widehat{P}^{(2)}_q$, $q = 1,2$.

Definition 6. Let's define that system (3) belongs to the class $C^{(2)}_{+.k}, k \in \{+,-\}$ if matrices $P^{(2)}(z) = P^{(2)}(te^{iv})$ are such that $(t,v) \in \Lambda^{(2)}_{+.k}(t_2)$, $k \in \{+,-\}$.

Let's introduce domains $G^{(2)}_{+.k}(t_2) = \{z = z(t,v): \ 0 < |z| < t_2, \ (t,v) \in \Lambda^{(2)}_{+.k}(t_2)\}$, $k \in \{+,-\}$.

Theorem. *Let $A(z)$ be an analytical matrix in the domain D_1 and $rang A(z) = p$ on the condition $z \in D_1$. Let system* (1) *may lead to the appearance* (2). *The vector-function $z^d H^{(2)}(z,Y,Y')$ has the property V_1 near the point* $(0,0,0)$. *Moreover, the following conditions are met for system* (3):

1) *the matrix $P^{(2)}(z)$ is analytical in the domain D_1 and has the property S_2 regarding the vector-function $\varphi = \varphi(z)$;*

2) *the vector-function* $F^{(2)} = F^{(2)}(z, Y)$ *is analytical in the domain* $D_1 \times G_1$, $F^{(2)}(0, 0) = 0$ *and has the property* M_2 *regarding the vector-function* $\varphi = \varphi(z)$*;*

3) *system* (3) *belongs to one of the classes* $C^{(2)}_{+.k}$, $k \in \{+, -\}$.

Then for each $k \in \{+, -\}$ *and for some* $t^* \in (0, t_2)$ *there are solutions of system* (1) $Y = Y(z)$, *which satisfy the initial conditions* $Y(z_0) = Y_0$ *for* $z_0 \in G^{(2)}_{+.k}(t^*)$, $Y_0 \in \{Y : |Y_j(z_0)| < \delta_j|\varphi_j(z_0)|, \ \delta_j > 0, \ j = \overline{1, p}\}$, *that are analytical in the domain* $G^{(2)}_{+.k}(t^*)$ *and for these solutions in this particular domain the estimates are fair:*

$$|Y_j(z)|^2 < \delta_j^2|\varphi_j(z)|^2, \ \ j = \overline{1, p}.$$

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