On Asymptotic Representations of One Class of Solutions of Second-Order Differential Equations

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Consider the differential equation

$$y'' = f(t, y, y'), \tag{1}$$

where $f : [a, \omega[\, \times \Delta_{Y_0} \times \Delta_{Y_1} \to \mathbf{R}$ is a continuous function, $-\infty < a < \omega \le +\infty$, Δ_{Y_i} $(i \in \{0, 1\})$ is a one-side neighborhood of Y_i and Y_i $(i \in \{0, 1\})$ is either 0 or $\pm\infty$. We assume that the numbers μ_i $(i = 0, 1)$ given by the formula

$$\mu_i = \begin{cases} 1 & \text{if either } Y_i = +\infty, \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is a right neighborhood of the point } 0, \\ -1 & \text{if either } Y_i = -\infty, \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is a left neighborhood of the point } 0 \end{cases}$$

satisfy the relations

$$\mu_0 \mu_1 > 0 \ \text{ for } \ Y_0 = \pm\infty \ \text{ and } \ \mu_0 \mu_1 < 0 \ \text{ for } \ Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of equation (1) defined in the left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \ \text{ for } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1). \tag{3}$$

Among the strictly monotonic, together with the derivatives of the first order, in some left neighborhood of ω of solutions of equation (1) we can single out only solutions admitting either representations of the form

$$y(t) = c_0 + o(1), \quad y(t) = \pi_\omega(t)[c_1 + o(1)] \ \text{ as } \ t \uparrow \omega, \tag{4}$$

where c_0, c_1 are nonzero real constants, or satisfying conditions (3).

The question of whether equation (1) has solutions with representations (4) can be, in general, solved using either for $\omega = +\infty$ a theorem from monograph [3, Ch. II, § 8, p. 207] or for $\omega \le +\infty$ ideas laid down in the work [1].

One of the classes of equation (1) solutions with properties (3) that admits asymptotic representations is the class of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions.

Definition 1. A solution y of equation (1) on the interval $[t_0, \omega[\, \subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if, in addition to (3), it satisfies the condition

$$\lim_{t \uparrow \omega} \frac{[y'(t)]^2}{y(t) y''(t)} = \lambda_0.$$

Depending on λ_0 these solutions have different asymptotic properties. For $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ in [2] such ratios

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}\,, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0 - 1}\,,$$

where

$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t-\omega & \text{if } \omega < +\infty, \end{cases}$$

are established.

Definition 2. We say that a function f satisfies condition $(FN)_{\lambda_0}$ for $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ if there exist a number $\alpha_0 \in \{-1,1\}$, a continuous function $p : [a,\omega[\, \to]0,+\infty[$ and twice continuously differentiable function $\varphi_0 : \Delta_{Y_0} \to]0,+\infty[\,$, satisfying the conditions

$$\varphi_0'(y) \neq 0, \quad \lim_{\substack{y\to Y_o \\ y\in\Delta_{Y_0}}} \varphi_0(y) = \varphi_0 \in \{0,+\infty\}, \quad \lim_{\substack{y\to Y_o \\ y\in\Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1, \tag{5}$$

such that, for arbitrary continuously differentiable functions $z_i : [a,\omega[\, \to \Delta_{Y_i}$ $(i=0,1)$ satisfying the conditions

$$\lim_{t\uparrow\omega} z_i(t) = Y_i \ \ (i=0,1),$$
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)z_0'(t)}{z_0(t)} = \frac{\lambda_0}{\lambda_0 - 1}\,, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)z_1'(t)}{z_1(t)} = \frac{1}{\lambda_0 - 1}\,,$$

one has representation

$$f(t,z_0(t),z_1(t)) = \alpha_0 p(t)\varphi_0(z_0(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega. \tag{6}$$

Note that the choice of α_0 and the functions p and φ_0 in Definition 2 depends on the choice of $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$. It is also obvious that the numbers μ_0, μ_1 determine the signs of any $P_\omega(Y_0,Y_1,\lambda_0)$-solution of equation (1) and its derivative in a left neighborhood of ω. Moreover, under condition $(FN)_{\lambda_0}$ sign of second derivative of any $P_\omega(Y_0,Y_1,\lambda_0)$-solution of equation (1) in a left neighborhood of ω coincides with the value α_0. Then taking into account (2), we have

$$\alpha_0\mu_1 > 0 \ \text{ for } \ Y_1 = \pm\infty \ \text{ and } \ \alpha_0\mu_1 < 0 \ \text{ for } \ Y_1 = 0. \tag{7}$$

We choose a number $b \in \Delta_{Y_0}$ such that the inequality

$$|b| < 1 \ \text{ for } \ Y_0 = 0, \quad b > 1 \ \ (b < -1) \ \text{ for } \ Y_0 = +\infty \ \ (Y_0 = -\infty)$$

is respected and put

$$\begin{cases} \Delta_{Y_0}(b) = [b, Y_0[& \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0, \\ \Delta_{Y_0}(b) =]Y_0, b] & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0. \end{cases}$$

Now we introduce auxiliary functions and notation as follows:

$$\Phi : \Delta_{Y_0}(b) \to \mathbb{R}, \quad \Phi(y) = \int_B^y \frac{ds}{\varphi_0(s)}\,, \quad B = \begin{cases} b & \text{if } \displaystyle\int_b^{Y_0} \frac{ds}{\varphi_0(s)} = \pm\infty, \\ Y_0 & \text{if } \displaystyle\int_b^{Y_0} \frac{ds}{\varphi_0(s)} = const, \end{cases}$$

$$Z = \lim_{y\to Y_0} \Phi(y) = \begin{cases} 0 & \text{if } B = Y_0, \\ +\infty & \text{if } B = b \text{ and } \mu_0\mu_1 > 0, \\ -\infty & \text{if } B = b \text{ and } \mu_0\mu_1 < 0, \end{cases} \quad \mu_2 = \begin{cases} 1 & \text{if } B = b, \\ -1 & \text{if } B = Y_0, \end{cases}$$

$$I(t) = \int_A^t \pi_\omega(\tau)p(\tau)\,d\tau, \quad A = \begin{cases} a & \text{if } \displaystyle\int_a^\omega \pi_\omega(\tau)p(\tau)\,d\tau = \pm\infty, \\ \omega & \text{if } \displaystyle\int_a^\omega \pi_\omega(\tau)p(\tau)\,d\tau = const. \end{cases}$$

Theorem 1. *Let $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ and let the function f satisfy condition $(FN)_{\lambda_0}$. Then, for the existence of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the differential equation* (1)*, it is necessary that the sign conditions* (2), (7),

$$\alpha_0\mu_0\lambda_0 > 0, \quad \mu_0\mu_1\lambda_0(\lambda_0-1)\pi_\omega(t) > 0, \quad \alpha_0\mu_2(\lambda_0-1)I(t) < 0 \ \textit{ for } \ t \in [a,\omega[$$

and

$$\alpha_0(\lambda_0-1)\lim_{t\uparrow\omega} I(t) = Z, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{\alpha_0(\lambda_0-1)\pi_\omega^2(t)p(t)\varphi_0(Y(t))}{Y(t)} = \frac{\lambda_0}{\lambda_0-1}$$

hold, where

$$Y(t) = \Phi^{-1}\big(\alpha_0(\lambda_0-1)I(t)\big).$$

Moreover, each solution of this kind admits the asymptotic representations

$$\frac{y'(t)}{\varphi_0(y(t))} = \alpha_0(\lambda_0-1)\pi_\omega(t)p(t)[1+o(1)], \quad \varphi_0'(y(t)) = -\frac{\lambda_0(1+o(1))}{(\lambda_0-1)I(t)} \ \textit{ as } \ t\uparrow\omega.$$

Remark 1. Asymptotic representations of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of equation (1) can be written explicitly

$$y(t) = Y(t)\Big(1 + \frac{o(1)}{H(t)}\Big), \quad y'(t) = \frac{\lambda_0}{\lambda_0-1}\,\frac{Y(t)}{\pi_\omega(t)}\,(1+o(1)),$$

where

$$H(t) = \frac{Y(t)\varphi'(Y(t))}{\varphi(Y(t))}\,.$$

References

[1] V. M. Evtukhov, Asymptotic properties of the solutions of a certain class of second-order differential equations. (Russian) *Math. Nachr.* **115** (1984), 215–236.

[2] V. M. Evtukhov, The asymptotic behavior of the solutions of one nonlinear second-order differential equation of the Emden–Fowler type. (Russian) *Dissertation Cand. Fiz.-Mat. Nauk: 01.01.02*, Odessa, Ukraine, 1998.

[3] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* (Russian) Nauka, Moscow, 1990.