Optimal Control Problems for Systems of Differential Equations with Impulse Action

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The problem of optimal control at a finite time interval for a system of differential equations with impulse action at fixed moments times and also the corresponding averaged system of ordinary differential equations are considered. The existence of optimal control of exact problem and averaged problem is proved, and also it is established that optimal control of averaged task carries out the approximate optimal synthesis of exact problem.

1 Introduction

In this paper, for the system of differential equations with impulse action at fixed moments of time, the problem of optimal control is considered:

$$\begin{gathered}
\dot{x}=\varepsilon\big[A(t,x)+B(t,x)u\big],\ \ t\neq t_i,\ \ i=1,2,\dots,i\Big(\frac{T}{\varepsilon}\Big),\ \ t\in\Big[0,\frac{T}{\varepsilon}\Big),\\
\triangle x\big|_{t=t_i}=\varepsilon I_i(x(t_i),v_i),\ \ i=1,2,\dots,i\Big(\frac{T}{\varepsilon}\Big),\\
x(0,u(0),v_i)=x_0,\ \ t_i<t_{i+1},
\end{gathered}\tag{1.1}$$

where $\varepsilon>0$ is a small parameter, $t\geq 0$, $T>0$ is some constant value, $x\in D$ is a phase n-dimensional vector, D is a region in R^n, $u\in U$ is a vector of control, U is convex and closed set in R^m, $0\in U$, $i(t)$ is the number of pulses on $[0,t)$: $t_1,t_2,\dots,t_n,\dots,t_{i(\frac{T}{\varepsilon})}$, and $t_n\to\infty$, $n\to\infty$; $v_i\in V$, $i=1,2,\dots,i(\frac{T}{\varepsilon})$, are impulse control vectors, V is a closed set in $\mathbb{R}^r$. With respect to the moments of impulsive action, we assume that there exists a constant $\widetilde{C}>0$ such that for $t\geq 0$,

$$i(t)\leq \widetilde{C}t.$$

A is an n-dimensional vector-function, B is an $n\times m$-dimensional matrix, $I_i(x,v)$ is an n-dimensional vector function.

Control $u=u(t)=(u_1(t),u_2(t),\dots,u_m(t))$ and $v=v_i=(v_{i1},v_{i2},\dots,v_{ir})$ will be considered admissible for problem (1.1), if (a1) $u(t) \in L_p(0, \frac{T}{\varepsilon})$ for some $p > 1$;

(a2) $u(t) \in U$ at $t \in [0, \frac{T}{\varepsilon}]$, almost everywhere;

a3) there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the solution $x(t, u, v)$ of the Cauchy problem (1.1) has defined by $t \in [0, \frac{T}{\varepsilon}]$, where ε_0 is independent of $u(t)$ and v_i;

(a4) $v_i \in V$;

(a5) for each sequence of control vectors $v_i \in V$ there exists the vector $v_0 \in V$ such that $v_i \to v_0$, $i \to \infty$, uniformly for all controls, that is, for arbitrary $\delta > 0$ there is a constant N_0, independent of v_i, v_0 and such that for all $i \geq N_0$ the inequality $|v_i - v_0| < \delta$ is satisfied.

It should be noted that condition a5) is obviously satisfied if there exists a sequence $\{a_i\}$ independent of v_i: $a_i \to 0$, $i \to \infty$, such that $|v_i - v_0| < a_i$.

We denote the set of valid controls by Ω.

By $|\cdot|$ we denote the norm of vector in Euclidean space, and through $\|\cdot\|$ we denote the norm of the matrix consistent with the norm of the vector. In this paper, the averaging method is applied to optimal control problems. The main role here is to justify the closeness of the solutions of the exact and average problems. This type of results for impulse systems was first obtained in [5] and further developed in the works of many scientists and applied to optimal control problems (see, for example, [4], where is comprehensive bibliography).

In works [3, 7, 8], another approach was developed to apply the averaging method to optimal control problems, where the control function was considered a fixed parameter when averaging. This approach had applied to the problems of optimal control of functional-differential equations in [2].

2 Formulation of the problem and the main result

The problem of optimal control to be solved in the work is to find such allowable controls $u(t)$ and v_i that minimize the functional

$$J_\varepsilon(u, v) = \varepsilon \int\limits_0^{\frac{T}{\varepsilon}} [C(t, x) + F(t, u)]\, dt + \varepsilon \sum_{0 \leq t_i < \frac{T}{\varepsilon}} \Psi_i(x(t_i), v_i),$$

here C, F, ψ_i are continuous in the set of variables of function, with $C \geq 0$, F and ψ_i satisfy the conditions:

$F(t, u)$ is defined for $t \geq 0$, $u \in U$, convex on u, and for some $a > 0$:

$$F(t, u) \geq a|u|^p, \quad \psi_i(t, v) \geq a|v|^p,$$

where $p > 1$ from condition $a1$) and for some $K > 0$ there exists $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0$ the inequality

$$\varepsilon \int\limits_0^{\frac{T}{\varepsilon}} F(t, 0)\, dt \leq K$$

holds.

With respect to system (1.1), we assume that the following conditions are fulfilled:

1.1) there are such $A_0(x)$, $B_0(x)$ and $C_0(x)$, for which uniformly over $x \in D$ the boundaries exist (averaging conditions):

$$\lim_{T\to\infty}\bigg|\frac{1}{T}\int\limits_0^T A(t,x)\,dt - A_0(x)\bigg| = 0,$$

$$\lim_{T\to\infty}\bigg|\frac{1}{T}\int\limits_0^T C(t,x)\,dt - C_0(x)\bigg| = 0,$$

$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T \|B(t,x) - B_0(x)\|^q\,dt = 0,$$

where q is determined from the condition $\frac{1}{p}+\frac{1}{q}=1$;

1.2) the vector function $A(t,x)$ and the matrix function $B(t,x)$ are defined, measurable by t for each x, the function $C(t,x)$ is defined and is continuous at $t \ge 0$, $x \in D$;

1.3) the functions $A(t,x)$, $B(t,x)$ and $C(t,x)$ are Lipschitz's functions on x with constant L in domain D;

1.4) the functions $I_i(x,v)$, $\psi_i(x,v)$, $i=1,2,\dots,i(t)$, are continuous on the set of variables;

1.5) the functions $\psi_i(x,v)$, $i=1,2,\dots,i(t)$, are bounded by the constant M at $t\ge 0$, $x\in D$, $v \in V$;

1.6) the functions $I_i(x,v)$, $\psi_i(x,v)$, $i=1,2,\dots,i(t)$, are Lipschitz's functions on x with constant L in the domain D and uniformly continuous on v in the domain of definition;

1.7) for the functions $A(t,x)$, $B(t,x)$, $C(t,x)$ and $I_i(x,v)$, $i=1,2,\dots,i(\frac{T}{\varepsilon})$, the conditions of linear growth are fulfilled, i.e., there is a constant $K>0$ such that for $t\ge 0$ and $x\in D$ the followings inequalities are fulfilled:

$$|A(t,x)| \le K(1+|x|),\ \ \|B(t,x)\| \le K(1+|x|),\ \ |I_i| \le K(1+|x|),\ \ |C(t,x)| \le K(1+|x|).$$

Let the averaging conditions also be satisfied:

1.8) uniformly for $x\in D$, $u\in U$, $v\in V$ there are boundaries:

$$\lim_{s\to\infty}\frac{1}{s}\sum_{0<t_i<s} I_i(x,v) = I_0(x,v),$$

$$\lim_{s\to\infty}\frac{1}{s}\sum_{0<t_i<s} \psi_i(x,v) = \psi_0(x,v).$$

Problem (1.1) on the interval $[0,\frac{T}{\varepsilon}]$ will correspond to the following averaged problem:

$$\begin{gathered}\dot y = \varepsilon\big[A_0(y) + B_0(y)\overline{u} + I_0(y,v_0)\big],\ \ t\in\Big[0,\frac{T}{\varepsilon}\Big), \\ y\big(0,\overline{u}(0),\overline{v}_i(0)\big) = x_0,\end{gathered} \tag{2.1}$$

where $\overline{u}$ is the allowable control of the averaging problem (2.1), that satisfies the same conditions as the allowable control of the exact problem (1.1), and v_0 for each v_i is selected from condition a5).

The set of admissible controls $(u(t), v_0)$ of problem (2.1) is denoted by $\overline{\Omega}$. The quality criterion of the problem of averaging is as follows:

$$\overline{J}_\varepsilon(\overline{u}, \overline{v}) = \varepsilon \int\limits_0^{\frac{T}{\varepsilon}} \big[C_0(y(t)) + F(t, \overline{u}) + \psi_0(y(t), v_0)\big]\, dt.$$

Let's denote

$$J_\varepsilon^* = \inf_{(u(t), v_i) \in \Omega} J_\varepsilon(u, v),$$
$$\overline{J}_\varepsilon^* = \inf_{(u(t), v_0) \in \Omega} \overline{J}_\varepsilon(\overline{u}, \overline{v}).$$

The purpose of this work is to prove for the problem of optimal control the following statement: for an arbitrary $\eta > 0$ there is $\varepsilon_0 = \varepsilon_0(\eta)$ such that for $\varepsilon < \varepsilon_0$ the inequality

$$\big|J_\varepsilon^* - J_\varepsilon(\overline{u}^*, v_0^*)\big| \le \eta$$

holds; $\overline{u}^*$, v_0^* is the optimal control pair for the problem of averaging, i.e., the optimal control of the problem of averaging is almost optimal for the exact one.

For the averaged system (2.1) we assume that the following condition is fulfilled:

(A) If the control $\overline{u}$ satisfies the estimate

$$\varepsilon \int\limits_0^{\frac{T}{\varepsilon}} |\overline{u}(t)|\, dt \le R,$$

where $R > 0$ does not depend on ε, $\overline{u}$, then there is $\varepsilon_0 = \varepsilon_0(R)$ such that for $0 < \varepsilon < \varepsilon_0$ the solution of the averaged Cauchy problem $y(t, \overline{u}, v_0)$ for $t \in [0, \frac{T}{\varepsilon}]$ lies in the region D together with some ρ-neighborhood, and ρ does not depend on ε, $\overline{u}$, v_0.

The following theorem holds.

Theorem. *Under conditions* 1.1)–1.7) *and condition* (A) *there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *the exact and averaged control problems have solutions, and for an arbitrary* $\eta > 0$ *there exists* $\varepsilon_1 = \varepsilon_1(\eta) \le \varepsilon_0$ *such that for* $0 < \varepsilon < \varepsilon_1$ *the inequality*

$$\big|J_\varepsilon^* - J_\varepsilon(\overline{u}^*, v_0^*)\big| \le \eta$$

is fulfilled, where $(\overline{u}^*, v_0^*)$ *is the optimal control of the averaging system.*

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