On Some Positive Solutions to Differential Equations with General Power-Law Nonlinearities

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1 Introduction

Consider solutions with positive initial data to differential equation with general power-law nonlinearity

$$y^{(n)}=p(x,y,y',\dots,y^{(n-1)})|y|^{k_0}|y'|^{k_1}\cdots|y^{(n-1)}|^{k_{n-1}}\operatorname{sgn}(y\,y'\cdots y^{(n-1)}), \tag{1.1}$$

with $n\geq 2$, positive real nonlinearity exponents $k_0,k_1,\dots,k_{n-1}$ and positive continuous in x and Lipschitz continuous in $u_0,u_1,\dots,u_{n-1}$ bounded function $p(u_0,u_1,\dots,u_{n-1})$.

The results on qualitative behavior and asymptotic estimates of positive increasing solutions for higher order nonlinear differential equations were obtained by I. T. Kiguradze and T. A. Chanturia in [9]. Questions on qualitative and asymptotic behavior of solutions to higher order Emden–Fowler differential equations ($k_1=\cdots=k_{n-1}=0$) were studied by I. V. Astashova in [1, 4–6].

In the case $n=2$ the results on qualitative behavior of solutions can be found in [10], and asymptotic behavior was studied in [11].

Equation (1.1) in the case $n=3$, $k_0>0$, $k_0\neq 1$, $k_1=k_2=0$, was studied by I. Astashova in [1, Chapters 6–8]. In particular, asymptotic classification of solutions to such equations was given in [3, 6], and proved in [2]. Qualitative properties of solutions in the case $n=3$, $k_0>0$, $k_1>0$, $k_2>0$ were studied in [12]. In this paper several results are generalized for higher order differential equations with general power-law nonlinearity.

For higher order differential equations, nonlinear with respect to derivatives of solutions, the asymptotic behavior of certain types of solutions was studied by V. M. Evtukhov, A. M. Klopot in [7, 8].

2 On the behavior of solutions

For equations of second- and third-order the following results on qualitative behavior of solutions with positive initial data were obtained.

Theorem 2.1 ([10]). *Suppose $n=2$ and $k_0+k_1>1$. Let the function $p(x,u,v)$ be continuous in x, Lipschitz continuous in u, v and satisfy the inequality $p(x,u,v)\geq m>0$. Then there exists a constant $\zeta=\zeta(m,k_0,k_1)$ such that any maximally extended solution $y(x)$ to equation (1.1), satisfying at some point x_0 the conditions $y(x_0)\geq 0$, $y'(x_0)=y_1>0$, has a finite right domain boundary x^* satisfying the estimate*

$$x^*-x_0<\zeta\,y_1^{-\frac{k_0+k_1-1}{k_0+1}}.$$

Theorem 2.2 ([12])**.** *Suppose $n = 3$ and $k_0 + k_1 + k_2 > 1$. Let the function $p(x, u, v, w)$ be continuous, Lipschitz continuous in u, v, w and satisfy the inequality $p(x, u, v, w) \ge m > 0$. Then there exists a constant $\psi = \psi(m, k_0, k_1, k_2)$ such that any maximally extended solution $y(x)$ to equation* (1.1)*, satisfying at some point x_0 the conditions $y(x_0) \ge 0$, $y'(x_0) \ge 0$, $y''(x_0) = y_2 > 0$, has a finite right domain boundary x^* satisfying the estimate*

$$x^* - x_0 < \psi\, y_2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}}.$$

Consider now the asymptotic behavior of solutions with positive initial data to second- and third order equations. For second order equation the result is also obtained for general form of $p(x, u, v)$ in [11].

Theorem 2.3 ([11])**.** *Suppose $n = 2$ and $k_0 + k_1 > 1$, $k_1 < 2$. Let $p(x, u, v) \equiv p_0 > 0$, and let x^* be a right boundary of the domain of an inextensible solution $y(x)$ to* (1.1) *satisfying at some point x_0 the conditions $y(x_0) > 0$, $y'(x_0) > 0$. Then*

$$y = C(x^* - x)^{-\alpha}(1 + o(1)), \quad x \to x^* - 0,$$

where

$$\alpha = \frac{2 - k_1}{k_0 + k_1 - 1} > 0, \quad C = \Big(\frac{|\alpha|^{1-k_1}|\alpha + 1|}{p_0}\Big)^{\frac{1}{k_0+k_1-1}}.$$

Theorem 2.4. *Suppose $n = 3$ and $k_0+k_1+k_2 > 1$, $k_2 < 1$, $k_1+2k_2 < 3$. Let $p(x, u, v, w) \equiv p_0 > 0$, and let x^* be a right boundary of the domain of an inextensible solution $y(x)$ to* (1.1) *satisfying at some point x_0 the conditions $y(x_0) > 0$, $y'(x_0) > 0$, $y''(x_0) > 0$. Then*

$$y = C(x^* - x)^{-\alpha}(1 + o(1)), \quad x \to x^* - 0,$$

where

$$\alpha = \frac{3 - k_1 - 2k_2}{k_0 + k_1 + k_2 - 1} > 0, \quad C = \Big(\frac{|\alpha|^{1-k_1-k_2}|\alpha + 1|^{1-k_2}|\alpha + 2|}{p_0}\Big)^{\frac{1}{k_0+k_1+k_2-1}}.$$

It turns out that it is possible to generalize the above results for the higher order equation (1.1). Denote

$$K = \sum_{i=0}^{n-1} k_i, \quad \varkappa = \sum_{i=1}^{n-1} i\, k_{n-1-i}.$$

Theorem 2.5. *Suppose $n \ge 2$, $K > 1$. Let the function $p(u_0, u_1, \dots, u_{n-1})$ be continuous in x, Lipschitz continuous in $u_0, \dots, u_{n-1}$ and satisfy the inequality*

$$p(x, y, y', \dots, y^{(n-1)}) \ge m > 0.$$

Then there exists a constant $\xi = \xi(n, m, k_0, \dots, k_{n-1})$ such that any maximally extended solution $y(x)$ to (1.1)*, satisfying at some point x_0 the conditions $y(x_0) > 0$, $y'(x_0) > 0, \dots, y^{(n-2)}(x_0) > 0$, $y^{(n-1)}(x_0) = y_{n-1} > 0$, has a finite right domain boundary x^* satisfying the estimate*

$$x^* - x_0 < \xi\, y_{n-1}^{-\frac{K-1}{\varkappa+1}}.$$

The following theorem states the existence of a solution in the form $y = C(x^* - x)^{-\alpha}$ to equation (1.1) with constant potential $p(x, y, y', \dots, y^{(n-1)}) \equiv (-1)^{n-1} p_0$.

Denote $\overline{\varkappa} = \sum\limits_{i=1}^{n-1} i\, k_i$.

Theorem 2.6. *Let $n \ge 2$, $p_0 > 0$ and $K > 1$. Then equation*

$$y^{(n)} = (-1)^{n-1} p_0 |y|^{k_0} |y'|^{k_1} \cdots |y^{(n-1)}|^{k_{n-1}} \operatorname{sgn}(y\, y' \cdots y^{(n-1)}) \tag{2.1}$$

has a solution $y = C(x^* - x)^{-\alpha}$, where $x^* < \infty$ is the right domain boundary,

$$C = \left(\frac{\prod\limits_{i=0}^{n-1} |\alpha + i|^{1 - \sum\limits_{i+1}^{n-1} k_i}}{p_0} \right)^{\frac{1}{K-1}}, \quad \alpha = \frac{n - \overline{\varkappa}}{K-1}\,.$$

Note that the higher order Emden–Fowler equation

$$y^{(n)} = p_0 |y|^k \operatorname{sgn} y, \quad n \ge 2, \quad k > 1, \quad p_0 > 0$$

for any $x^* \in \mathbb{R}$ has the solution $y = C(x^* - x)^{-\alpha}$ with

$$\alpha = \frac{n}{k-1}\,, \quad C = \left(\frac{\alpha(\alpha+1)\cdots(\alpha+n-2)(\alpha+n-1)}{p_0} \right)^{\frac{1}{k-1}},$$

which corresponds to the result obtained in Theorem 2.6 with $k_1 = \cdots = k_{n-1} = 0$ (see [1, **5.1**]). The existence of solutions to equation (1.1) which is equivalent to $C(x^* - x)^{-\alpha}$ as $x \to x^* - 0$ in general case is an open problem. For $n = 2$ this problem was solved in [11], and for $n \ge 3$, $k_1 = \cdots = k_{n-1} = 0$ it was solved in [1, Chapter 5] and [4, 6].

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References

[1] I. V. Astashova, Qualitative properties of solutions to quasilinear ordinary differential equations. (Russian) In: Astashova I. V. (Ed.) *Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis*, pp. 22–290, UNITY-DANA, Moscow, 2012.

[2] I. V. Astashova, On asymptotic classification of solutions to nonlinear third-and fourth-order differential equations with power nonlinearity. *Vestnik MGTU im. NE Baumana. Ser. Estestvennye nauki*, 2015, 3–25.

[3] I. V. Astashova, On asymptotic classification of solutions to nonlinear regular and singular third- and fourth-order differential equations with power nonlinearity. *Differential and difference equations with applications*, 191–203, Springer Proc. Math. Stat., 164, *Springer, [Cham]*, 2016.

[4] I. V. Astashova, On Kiguradze's problem on power-law asymptotic behavior of blow-up solutions to Emden-Fowler type differential equations. *Georgian Math. J.* 24 (2017), no. 2, 185–191.

[5] I. V. Astashova, On asymptotic behaviour of one-dimensional Schrödinger equation with complex coefficients. *J. Nat. Geom.* **19** (2001), no. 1-2, 39–52.

[6] I. V. Astashova, Asymptotic behavior of singular solutions of Emden–Fowler type equations. (Russian) *Differ. Uravn.* **55** (2019), no. 5, 597–606; translation in *Differ. Equ.* **55** (2019), no. 5, 581–590.

[7] V. M. Evtukhov, A class of monotone solutions of an nth-order nonlinear differential equation of Emden–Fowler type. (Russian) *Soobshch. Akad. Nauk Gruzii* **145** (1992), no. 2, 269–273.

[8] V. M. Evtukhov and A. M. Klopot, Asymptotic representations for some classes of solutions of ordinary differential equations of order n with regularly varying nonlinearities. (Ukrainian) *Ukr. Mat. Zh.* **65** (2013), no. 3, 354–380; translation in *Ukr. Math. J.* **65** (2013), no. 3, 393–422.

[9] I. T. Kiguradze and T. A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. Mathematics and its Applications (Soviet Series), 89. Kluwer Academic Publishers Group, Dordrecht, 1993.

[10] T. Korchemkina, On the behavior of solutions to second-order differential equation with general power-law nonlinearity. *Mem. Differ. Equ. Math. Phys.* **73** (2018), 101–111.

[11] T. A. Korchemkina, Asymptotic behavior of unbounded solutions of second-order differential equations with general nonlinearities. *Translation of Tr. Semin. im. I. G. Petrovskogo* No. 32 (2019), 239–256; translation in *J. Math. Sci. (N.Y.)* **244** (2020), no. 2, 267–277.

[12] T. Korchemkina, On the behavior of solutions with positive initial data to third order differential equations with general power-law nonlinearities. *Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2019, Tbilisi, Georgia, December 7-9*, pp. 112–117;
`http://www.rmi.ge/eng/QUALITDE-2019/Korchemkina_workshop_2019.pdf`.