On One-Dimensional Nonlinear Integro-Differential System with Source Terms

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The process of electromagnetic field propagation into a substance, its mathematical modeling, investigation, and numerical solution belong to one of the most important tasks in applied mathematics. As a rule, this process is accompanied by the release of thermal energy, which causes changes in the permeability of the medium and affects the diffusion process since the coefficient of conductivity of the medium significantly depends on temperature. Mathematical simulation of the mentioned process, like many other applied problems, results in nonlinear partial differential and integro-differential equations and systems of those equations. In a quasistationary case the corresponding system of the Maxwell equations has the following form [12]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m characterize the heat capacity and electrical conductivity of the medium. Equation (1) describes the propagation of the magnetic field in the medium whereas equation (2) expresses a change of the temperature due to the Joule heating. Assume that coefficients of thermal heat capacity and electrical conductivity of the substance depending on temperature. In this case, as it is shown in [3], system (1), (2) can be reduced to the following nonlinear parabolic type integro-differential model

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\bigg[a\bigg(\int_0^t |\operatorname{rot} H|^2\, d\tau\bigg) \operatorname{rot} H\bigg]. \tag{3}$$

Let us note that the above-mentioned integro-differential model (3) is complex and only particular classes are investigated (see, for example, [1–11, 13–15, 17, 18] and the references therein). Consider the case when all three components of the magnetic field vector are functions of time and one spatial variable $H_i = H_i(x,t)$, $i = 1, 2, 3$. Thus, in this case we have:

$$\operatorname{rot} H = \Big(0, -\frac{\partial H_3}{\partial x}, \frac{\partial H_2}{\partial x}\Big),$$

$$\operatorname{rot}(a(S) \operatorname{rot} H) = \bigg(0, -\frac{\partial}{\partial x}\Big(a(S)\,\frac{\partial H_2}{\partial x}\Big), -\frac{\partial}{\partial x}\Big(a(S)\,\frac{\partial H_3}{\partial x}\Big)\bigg),$$

where

$$S(x,t) = \int_0^t \bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2 + \Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\, d\tau,$$

and system (3) takes the following form:

$$
\begin{gathered}
\frac{\partial H_1}{\partial t} = 0, \\
\frac{\partial H_2}{\partial t} - \frac{\partial}{\partial x}\left[a\bigg(\int_0^t \bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2 + \Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\,d\tau\bigg)\,\frac{\partial H_2}{\partial x}\right] = 0, \\
\frac{\partial H_3}{\partial t} - \frac{\partial}{\partial x}\left[a\bigg(\int_0^t \bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2 + \Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\,d\tau\bigg)\,\frac{\partial H_3}{\partial x}\right] = 0.
\end{gathered}
\tag{4}
$$

Our goal is to study the convergence of the finite difference scheme for the following initial-boundary value problem posed for the nonlinear integro-differential system (4) with source terms and known right-hand sides:

$$
\begin{gathered}
\frac{\partial H_1}{\partial t} + g_1(H_1) = f_1, \\
\frac{\partial H_2}{\partial t} - \frac{\partial}{\partial x}\left[a\bigg(\int_0^t \bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2 + \Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\,d\tau\bigg)\,\frac{\partial H_2}{\partial x}\right] + g_2(H_2) = f_2, \\
\frac{\partial H_3}{\partial t} - \frac{\partial}{\partial x}\left[a\bigg(\int_0^t \bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2 + \Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\,d\tau\bigg)\,\frac{\partial H_3}{\partial x}\right] + g_3(H_3) = f_3,
\end{gathered}
\tag{5}
$$

$$
H_2(0,t) = H_2(1,t) = H_3(0,t) = H_3(1,t) = 0, \;\; t \geq 0, \tag{6}
$$

$$
H_1(x,0) = H_{10}(x), \quad H_2(x,0) = H_{20}(x), \quad H_3(x,0) = H_{30}(x), \;\; x \in [0,1], \tag{7}
$$

where H_{i0}, g_i, f_i, $i = 1,2,3$ are given functions and g_i are monotonically increased and positively defined functions.

Due to the fact that the last two equations of system (5) are strongly connected to each other, we will consider these equation jointly, whereas the first equation will be considered independently.

Let us correspond the finite difference scheme for problem (5)–(7). On $[0,1]\times[0,T]$ let us introduce a net with mesh points denoted by $(x_i,t_j) = (ih, j\tau)$, where $i = 0,1,\dots,M$; $j = 0,1,\dots,N$ with $h = 1/M$, $\tau = T/N$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i,t_j) is designed by (u_i^j, v_i^j, w_i^j) and the exact solution to problem (5)–(7)) by $(H_{1i}^j, H_{2i}^j, H_{3i}^j)$. We will use the following known notations [16] of forward and backward derivatives:

$$
r_{x,i}^j = \frac{r_{i+1}^j - r_i^j}{h}\,, \quad r_{\overline{x},i}^j = \frac{r_i^j - r_{i-1}^j}{h}\,, \quad r_{t,i}^j = \frac{r_i^{j+1} - r_i^j}{\tau}\,,
$$

and inner products and corresponding norms:

$$
\begin{gathered}
(r^j, y^j) = h\sum_{i=1}^{M-1} r_i^j y_i^j, \quad (r^j, y^j] = h\sum_{i=1}^{M} r_i^j y_i^j, \\
\|r^j\| = (r^j, r^j)^{1/2}, \quad \|r^j]| = (r^j, r^j]^{1/2}.
\end{gathered}
$$

For problem (5)–(7) let us consider the following finite difference scheme:

$$
\begin{gathered}
\frac{u_i^{j+1}-u_i^j}{\tau}+g_1(u_i^{j+1})=f_{1,i}^j,\\
\frac{v_i^{j+1}-v_i^j}{\tau}-\bigg\{a\bigg(\tau\sum_{k=1}^{j+1}\big[(v_{\overline{x},i}^k)^2+(w_{\overline{x},i}^k)^2\big]\bigg)v_{\overline{x},i}^{j+1}\bigg\}_{x,i}+g_2\left(v_i^{j+1}\right)=f_{2,i}^j,\\
\frac{w_i^{j+1}-w_i^j}{\tau}-\bigg\{a\bigg(\tau\sum_{k=1}^{j+1}\big[(v_{\overline{x},i}^k)^2+(w_{\overline{x},i}^k)^2\big]\bigg)w_{\overline{x},i}^{j+1}\bigg\}_{x,i}+g_3\left(w_i^{j+1}\right)=f_{3,i}^j,\\
i=1,2,\dots,M-1;\ \ j=0,1,\dots,N-1,\\
v_0^j=v_M^j=w_0^j=w_M^j=0,\ \ j=0,1,\dots,N,\\
u_i^0=H_{10,i},\quad v_i^0=H_{20,i},\quad v_i^0=H_{30,i},\ \ i=0,1,\dots,M.
\end{gathered}
\tag{8}
$$

Multiplying equations in (8) scalarly by u^{j+1}, v^{j+1} and w^{j+1}, respectively, it is not difficult to get the inequalities:

$$
\|u^n\|<C,\quad \|v^n\|^2+\sum_{j=1}^n\|v_{\overline{x}}^j]|^2\tau<C,\quad \|w^n\|^2+\sum_{j=1}^n\|w_{\overline{x}}^j]|^2\tau<C,\ \ n=1,2,\dots,N, \tag{9}
$$

where here and below C is a positive constant independent from τ and h.

The a priori estimates (9) guarantee the stability of scheme (8). The main statement of this note can be stated as follows.

Theorem. *If $a=a(S)\geq a_0=Const>0$, $a'(S)\geq 0$, $a''(S)\leq 0$ and g_i, $i=1,2,3$ are positively defined and monotonically increased functions, and problem* (5)–(7) *has a sufficiently smooth solution, then the solution of the difference scheme* (8) *tends to the solution of the continuous problem* (5)–(7) *as $\tau\to 0$, $h\to 0$ and the following estimates are true:*

$$
\|u^j-H_1^j\|\leq C(\tau),\quad \|v^j-H_2^j\|\leq C(\tau+h),\quad \|w^j-H_3^j\|\leq C(\tau+h).
$$

We have carried out numerous numerical experiments for problem (5)–(7) with different kind of right hand sides and initial-boundary conditions. Results of numerical experiments confirmed findings in the above-stated theorem.

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