# On a Dirichlet Type Boundary Value Problem for a Class of Linear Partial Differential Equations

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Let $\Omega=(0,\omega_1)\times(0,\omega_2)\times(0,\omega_3)$ be an open rectangular box, and let $E$ be an *orthogonally convex* piecewise smooth domain inscribed in $\Omega$.

A set $G\in\mathbb{R}^n$ is defined to be ***orthogonally convex*** if, for every line $L$ that is parallel to one of standard basis vectors, the intersection of $G$ with $L$ is empty, a point, or a single segment.

In the domain $E$ consider the boundary value problem

$$u^{(2,2,2)}=\sum_{\alpha<\mathbf{2}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}), \tag{1}$$

$$u\,\nu_1\big|_{\partial E}=\nu_1(\mathbf{x})\psi_1(\mathbf{x}),\quad u^{(2,0,0)}\,\nu_2\big|_{\partial E}=\nu_2(\mathbf{x})\psi_2(\mathbf{x}),\quad u^{(2,2,0)}\,\nu_3\big|_{\partial E}=\nu_3(\mathbf{x})\psi_3(\mathbf{x}). \tag{2}$$

Here $\mathbf{x}=(x_1,x_2,x_3)$, $\mathbf{2}=(2,2,2)$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)$ is a multi-index,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x})=\frac{\partial^{\alpha_1+\alpha_2+\alpha_3}u(\mathbf{x})}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\partial x_3^{\alpha_3}}\,,$$

$\partial E$ is the boundary of $E$, and $\boldsymbol{\nu}(\mathbf{x})=(\nu_1(\mathbf{x}),\nu_2(\mathbf{x}),\nu_2(\mathbf{x}))$ is the outward unit normal vector at point $\mathbf{x}\in\partial E$, $p_{\boldsymbol{\alpha}}\in C(\overline{E})$ $(\boldsymbol{\alpha}<\mathbf{2})$, $q\in C(\overline{E})$, $\psi_i\in C^{2,2,2}(\overline{E})$ and $\overline{E}$ is the closure of $E$.

By a solution of problem (1),(2) we understand a *classical* solution, i.e., a function $u\in C^{2,2,2}(E)\cap C^{2,2,0}(\overline{E})$ satisfying equation (1) and the boundary conditions (2) everywhere in $E$ and $\partial E$, respectively.

$C^{2,2,2}(E)$ is the space of continuous functions $u:E\to\mathbb{R}$ having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha}\le\mathbf{2})$.

Throughout the paper the following notations will be used.

$\mathbf{0}=(0,0,0)$, $\mathbf{1}=(1,1,1)$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)<\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\alpha_i\le\beta_i$ $(i=1,2,3)$ and $\boldsymbol{\alpha}\ne\boldsymbol{\beta}$.

$\boldsymbol{\alpha}=(\alpha_1,\alpha_2,\alpha_3)\le\boldsymbol{\beta}=(\beta_1,\beta_2,\beta_3)\iff\boldsymbol{\alpha}<\boldsymbol{\beta}$, or $\boldsymbol{\alpha}=\boldsymbol{\beta}$.

$\|\boldsymbol{\alpha}\|=|\alpha_1|+|\alpha_2|+|\alpha_3|$.

$\boldsymbol{\Upsilon}_{\mathbf{2}}=\big\{\boldsymbol{\alpha}<\mathbf{2}:\ \alpha_i=2 \text{ for some } i\in\{1,2,3\}\big\}$.

$\boldsymbol{O}_{\mathbf{2}}=\big\{\boldsymbol{\alpha}<\mathbf{2}:\ \|\boldsymbol{\alpha}\| \text{ is odd}\big\}$.

$\operatorname{supp}\boldsymbol{\alpha}=\{i\mid\ \alpha_i>0\}$.

$\mathbf{x}_{\boldsymbol{\alpha}}=(\chi(\alpha_1)x_1,\chi(\alpha_2)x_2,\chi(\alpha_3)x_3)$, where $\chi(\alpha)=0$ if $\alpha=0$, and $\chi(\alpha)=1$ if $\alpha>0$.
$\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(x_{i_1},\dots,x_{i_l})$, where $\{i_1,\dots,i_l\}=\operatorname{supp}\boldsymbol{\alpha}$.

$\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}=\mathbf{x}-\mathbf{x}_{\boldsymbol{\alpha}}$.

$$f^+(z)=\frac{f(z)+|f(z)|}{2}\,,\quad f^-(z)=\frac{|f(z)|-f(z)}{2}\,.$$

$H(f)(\mathbf{x})$ is the Hessian matrix of function $f$ at point $\mathbf{x}$.

Along with problem (1), (2) consider the corresponding homogeneous problem

$$u^{(2,2,2)} = \sum_{\alpha<\mathbf{2}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{$1_0$}$$

$$u\,\nu_1\big|_{\partial E} = 0, \quad u^{(2,0,0)}\,\nu_2\big|_{\partial E} = 0, \quad u^{(2,2,0)}\,\nu_3\big|_{\partial E} = 0. \tag{$2_0$}$$

Two-dimensional versions of problem (1), (2) were studied in [1–4]. The case of a characteristic rectangle was considered [1] and [2]. In [3] and [4] two-dimensional problems were considered in a orthogonally convex smooth domains.

Orthogonal convexity and smoothness of a domain are essential requirements and cannot be relaxed. Examples attesting the paramount importance of orthogonal convexity and smoothness of a domain were introduced in Remarks 1 and 2 of [4]. Similar examples can be easily constructed for the three-dimensional case.

Characteristic rectangles were the only admissible piecewise smooth domains for two-dimensional problems. In the three-dimensional case admissible piecewise smooth domains consist of characteristic rectangular boxes and right cylinders with an orthogonally convex smooth base.

We study problem (1), (2) in the following three cases: characteristic rectangular box; a right cylinder with an orthogonally convex smooth base; an *orthogonally convex* smooth domain.

It is not difficult to show that the problem

$$u^{(2,2,2)} = 0,$$
$$u\,\nu_1\big|_{\partial E} = \nu_1(\mathbf{x})\psi_1(\mathbf{x}), \quad u^{(2,0,0)}\,\nu_2\big|_{\partial E} = \nu_2(\mathbf{x})\psi_2(\mathbf{x}), \quad u^{(2,2,0)}\,\nu_3\big|_{\partial E} = \nu_3(\mathbf{x})\psi_3(\mathbf{x})$$

is uniquely solvable in all three aforementioned cases. Consequently, without loss of generality, problem (1), (2) can always be reduced to the problem with the zero boundary conditions.

Due to this fact, for the sake of technical simplicity, all results will be formulated for problem (1), ($2_0$).

**Case I: Characteristic Rectangular Box.** Let $E = \Omega$. For the rectangular box $\Omega$ the boundary conditions ($2_0$) receive the form

$$u(\sigma\,\omega_1, x_1, x_2) = 0, \quad u^{(2,0,0)}(x_1, \sigma\,\omega_2, x_2) = 0, \quad u^{(2,2,0)}(x_1, x_2, \sigma\,\omega_3) = 0 \ \ (\sigma = 0, 1).$$

It is easy to see that the latter conditions are equivalent to the following ones

$$u(\sigma\,\omega_1, x_1, x_2) = 0, \quad u(x_1, \sigma\,\omega_2, x_2) = 0, \quad u(x_1, x_2, \sigma\,\omega_3) = 0 \ \ (\sigma = 0, 1). \tag{3}$$

**Theorem 1.** *Let*

$$p_{\boldsymbol{\alpha}}(\mathbf{x}) \equiv p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) \ \ \textit{if} \ \ \boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{2}} \cap \boldsymbol{O}_{\mathbf{2}},$$

*and let the following inequalities hold:*

$$p_{220}(\mathbf{x}) \equiv p_{220}(x_3) > -\frac{\pi^2}{\omega_3^2}, \quad p_{202}(\mathbf{x}) \equiv p_{202}(x_2) > -\frac{\pi^2}{\omega_2^2}, \quad p_{022}(\mathbf{x}) \equiv p_{022}(x_1) > -\frac{\pi^2}{\omega_1^2}, \tag{4}$$

$$p_{220}^{-}(x_3)\,\frac{\omega_3^2}{\pi^2} + p_{202}^{-}(x_2)\,\frac{\omega_2^2}{\pi^2} + p_{200}^{+}(\mathbf{x})\,\frac{\omega_2^2\omega_3^2}{\pi^4} + \big|p_{211}(\mathbf{x})\big|\,\frac{\omega_2\omega_3}{\pi^2} < 1, \tag{5}$$

$$p_{220}^{-}(x_3)\,\frac{\omega_3^2}{\pi^2} + p_{022}^{-}(x_1)\,\frac{\omega_1^2}{\pi^2} + p_{020}^{+}(\mathbf{x})\,\frac{\omega_1^2\omega_3^2}{\pi^4} + \big|p_{121}(\mathbf{x})\big|\,\frac{\omega_1\omega_3}{\pi^2} < 1, \tag{6}$$

$$p_{202}^{-}(x_2)\,\frac{\omega_2^2}{\pi^2} + p_{022}^{-}(x_1)\,\frac{\omega_1^2}{\pi^2} + p_{002}^{+}(\mathbf{x})\,\frac{\omega_1^2\omega_2^2}{\pi^4} + \big|p_{112}(\mathbf{x})\big|\,\frac{\omega_1\omega_2}{\pi^2} < 1. \tag{7}$$

*Then problem* (1), (3) *has the Fredholm property, i.e.:*

(i) *problem* $(1_0),(3)$ *has a finite dimensional space of solutions;*

(ii) *problem* $(1),(3)$ *is uniquely solvable if and only if problem* $(1_0),(3)$ *has only the trivial solution.*

*Furthermore, every solution of problem* $(1),(3)$ *belongs to* $C^{2,2,2}(\overline{\Omega})$.

**Remark 1.** The strict inequalities (4)–(7) are sharp and cannot be replaced by nonstrict ones. Violation of at least one of the above inequalities can lead to the loss of the Fredholm property of problem $(1),(2)$. To verify this, consider the problem

$$u^{(2,2,2)}=(-1)^{\|\boldsymbol{\alpha}\|}u^{(2\boldsymbol{\alpha})}+u-\sin x_1\sin x_2\sin x_3\,q(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}), \tag{8}$$

$$u(\sigma\,\pi,x_1,x_2)=0,\quad u(x_1,\sigma\,\pi,x_2)=0,\quad u(x_1,x_2,\sigma\,\pi)=0\ \ (\sigma=0,1) \tag{9}$$

in the domain $E=(0,\pi)\times(0,\pi)\times(0,\pi)$. Here $\mathbf{0}<\boldsymbol{\alpha}<\mathbf{1}$, and $q$ is an arbitrary ***non-differentiable*** continuous function. The problem satisfies all of the inequalities (4)–(7) except the one for the coefficient $p_{2\boldsymbol{\alpha}}$: instead of $(-1)^{\|\boldsymbol{\alpha}\|}p_{2\boldsymbol{\alpha}}>1$ we have $(-1)^{\|\boldsymbol{\alpha}\|}p_{2\boldsymbol{\alpha}}=1$. As a result, problem $(8),(9)$ does not have the Fredholm property. Indeed, despite the fact that the homogeneous problem

$$u^{(2,2,2)}=(-1)^{\|\boldsymbol{\alpha}\|}u^{(2\boldsymbol{\alpha})}+u,$$
$$u(\sigma\,\pi,x_1,x_2)=0,\quad u(x_1,\sigma\,\pi,x_2)=0,\quad u(x_1,x_2,\sigma\,\pi)=0\ \ (\sigma=0,1)$$

has only the trivial solution, problem $(8),(9)$ has the unique weak solution

$$u(x)=\sin x_1\sin x_2\sin x_3\,q(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}),$$

which is not a classical solution due to ***non-differentiability*** of the function $q$.

Consider the equation

$$u^{(2,2,2)}=\sum_{\boldsymbol{\alpha}<\mathbf{1}}p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})u^{(2\boldsymbol{\alpha})}+\sum_{\alpha\in\boldsymbol{O_2}}p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}). \tag{10}$$

**Corollary 1.** *Let*

$$(-1)^{\|\boldsymbol{\alpha}\|}p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})\geq 0\ \ \textit{for}\ \ \boldsymbol{\alpha}<\mathbf{1}. \tag{11}$$

*Then problem* $(10),(3)$ *is uniquely solvable.*

**Case II: Right Cylinder.** Let $E=\{(x_1,x_2,x_3)\in\Omega:\ (x_1,x_2)\in G,\ x_3\in(0,\omega_3)\}$, where $G$ is an *orthogonally convex* open domain with $C^2$ boundary inscribed in the rectangle $(0,\omega_1)\times(0,\omega_2)$, i.e.,

$$\begin{aligned}G=\big\{(x_1,x_2)\in\Omega:\ x_1\in(0,\omega_1),\ x_2\in(\gamma_1(x_1),\gamma_2(x_1))\big\}&\\ =\big\{(x_1,x_2)\in\Omega:\ x_2\in(0,\omega_2),\ x_1\in(\eta_1(x_2),\eta_2(x_2))\big\},&\end{aligned} \tag{12}$$

and $\gamma_i\in C([0,\omega_1])\cap C^2((0,\omega_1))$, $\eta_i\in C([0,\omega_2])\cap C^2((0,\omega_2))$ $(i=1,2)$.

In the right cylinder $E$ consider the following equations

$$\begin{aligned}u^{(2,2,2)}&=p_{220}(x_3)u^{(2,2,0)}+p_{202}(x_2)u^{(2,0,2)}+p_{200}(\mathbf{x})u^{(2,0,0)}+p_{020}(\mathbf{x})u^{(0,2,0)}+p_{002}(x_2,x_3)u^{(0,0,2)}\\&\quad+\sum_{\alpha\leq\mathbf{1}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}),\end{aligned} \tag{13}$$

$$\begin{aligned}u^{(2,2,2)}&=p_{220}(x_3)u^{(2,2,0)}+p_{202}(x_2)u^{(2,0,2)}+p_{022}(x_1)u^{(0,2,2)}\\&\quad+p_{200}(\mathbf{x})u^{(2,0,0)}+p_{020}(x_3)u^{(0,2,0)}+p_{002}(x_2)u^{(0,0,2)}+p_{000}(x_2,x_3)u+q(\mathbf{x})\end{aligned} \tag{14}$$

and

$$\begin{aligned} u^{(2,2,2)} &= p_{220}(x_3)u^{(2,2,0)} + p_{202}(x_2)u^{(2,0,2)} + p_{022}(x_1)u^{(0,2,2)} \\ &\quad + p_{200}(\mathbf{x})u^{(2,2,0)} + p_{020}(x_3)u^{(0,2,0)} + p_{002}(x_2)u^{(0,0,2)} \\ &\quad + \sum_{\alpha\leq\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}). \end{aligned} \tag{15}$$

In view of (12), conditions $(2_0)$ receive the form

$$\begin{aligned} &u(\zeta_i(x_2,x_3),x_2,x_3) = 0, \quad u^{(2,0,0)}(x_1,\eta_i(x_1,x_3),x_3) = 0, \\ &\qquad\qquad u^{(2,2,0)}(x_1,x_2,\gamma_i(x_1,x_2)) = 0 \ \ (i=1,2). \end{aligned} \tag{16}$$

**Theorem 2.** *Let the following inequalities hold:*

$$p_{220}(x_3) \geq 0, \quad p_{202}(x_2) \geq 0, \quad p_{200}(\mathbf{x}) \leq 0, \quad p_{020}(\mathbf{x}) \leq 0, \quad p_{002}(x_2,x_3) \leq 0.$$

*Then problem* (13), (16) *has the Fredholm property.*

**Theorem 3.** *Let $G$ be a* ***convex*** *domain, i.e.,*

$$(-1)^{i-1}\gamma_i''(x_1) \geq 0 \ \ \textit{for} \ \ x_1 \in (0,\omega_1) \ \ (i=1,2) \tag{17}$$

*and*

$$(-1)^{i-1}\eta_i''(x_2) \geq 0 \ \ \textit{for} \ \ x_2 \in (0,\omega_2) \ \ (i=1,2), \tag{18}$$

*and let*

$$p_{220}(x_3) \geq 0, \quad p_{202}(x_2) \geq 0, \quad p_{202}(x_1) \geq 0, \tag{19}$$

$$\begin{gathered} p_{200}(\mathbf{x}) \leq 0, \quad p_{020}(x_3) \leq 0, \quad p_{002}(x_2) \leq 0, \\ p_{000}(x_2,x_3) \geq 0. \end{gathered} \tag{20}$$

*Then problem* (14), (16) *is uniquely solvable.*
*Furthermore, if $G$ is* ***strongly convex****, i.e.,*

$$(-1)^{i-1}\gamma_i''(x_1) > 0 \ \ \textit{for} \ \ x_1 \in (0,\omega_1) \ \ (i=1,2) \tag{21}$$

*and*

$$(-1)^{i-1}\eta_i''(x_2) > 0 \ \ \textit{for} \ \ x_2 \in (0,\omega_2) \ \ (i=1,2), \tag{22}$$

*then the solution of problem* (14), (16) *belongs to* $C^{2,2,2}(\overline{E})$.

**Corollary 2.** *Let inequalities* (17)–(20) *hold. Then problem* (15), (16) *has the Fredholm property. Furthermore, if inequalities* (21) *and* (22) *hold, then every solution of problem* (15), (16) *belongs to* $C^{2,2,2}(\overline{E})$.

**Case III: Smooth Domain.** Let $E$ be an *orthogonally convex* open domain with $C^{2,2}$ boundary inscribed in the characteristic box $\Omega$, i.e.,

$$\begin{aligned} E &= \big\{(x_1,x_2,x_3) \in \Omega : \ (x_1,x_2) \in G_{12}, \ x_3 \in (\gamma_1(x_1,x_2),\gamma_2(x_1,x_2))\big\} \\ &= \big\{(x_1,x_2,x_3) \in \Omega : \ (x_1,x_3) \in G_{13}, \ x_2 \in (\eta_1(x_1,x_3),\eta_2(x_1,x_3))\big\} \\ &= \big\{(x_1,x_2,x_3) \in \Omega : \ (x_2,x_3) \in G_{13}, \ x_1 \in (\zeta_1(x_2,x_3),\zeta_2(x_2,x_3))\big\}, \end{aligned} \tag{23}$$

where $\gamma_i \in C(\overline{G}_{12}) \cap C^{2,2}(G_{12})$, $\eta_i \in C(\overline{G}_{13}) \cap C^{2,2}(G_{13})$, $\zeta_i \in C(\overline{G}_{23}) \cap C^{2,2}(G_{23})$ $(i=1,2)$, and $G_{12}$, $G_{13}$ and $G_{23}$ are orthogonally convex smooth open domains inscribed in $(0,\omega_1)\times(0,\omega_2)$, $(0,\omega_1)\times(0,\omega_3)$ and $(0,\omega_2)\times(0,\omega_3)$, respectively.

In the domain $E$ consider the following equations:

$$u^{(2,2,2)} = p_{220}(\mathbf{x})u^{(2,2,0)} + p_{200}(x_1,x_3)u^{(2,0,0)} + \sum_{\alpha\leq\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{24}$$

$$\begin{aligned} u^{(2,2,2)} &= p_{220}(x_3)u^{(2,2,0)} + p_{202}(x_2)u^{(2,0,2)} \\ &\quad + p_{200}(\mathbf{x})u^{(2,0,0)} + p_{020}(x_3)u^{(0,2,0)} + p_{002}(x_2)u^{(0,0,2)} + p_{000}(x_2,x_3)u + q(\mathbf{x}) \end{aligned} \tag{25}$$

and

$$\begin{aligned} u^{(2,2,2)} &= p_{220}(x_3)u^{(2,2,0)} + p_{202}(x_2)u^{(2,0,2)} + p_{200}(\mathbf{x})u^{(2,2,0)} + p_{020}(x_3)u^{(0,2,0)} + p_{002}(x_2)u^{(0,0,2)} \\ &\quad + \sum_{\alpha\leq\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}). \end{aligned} \tag{26}$$

In view of (23), conditions $(2_0)$ receive the form

$$\begin{aligned} u(\zeta_i(x_2,x_3),x_2,x_3) = 0, \quad & u^{(2,0,0)}(x_1,\eta_i(x_1,x_3),x_3) = 0, \\ & u^{(2,2,0)}(x_1,x_2,\gamma_i(x_1,x_2)) = 0 \;\; (i=1,2). \end{aligned} \tag{27}$$

**Theorem 4.** *Let the following inequalities hold:*

$$p_{220}(\mathbf{x}) \geq 0,$$
$$p_{200}(x_1,x_3) \leq 0.$$

*Then problem* (24), (27) *has the Fredholm property.*

**Theorem 5.** *Let $E$ be a **convex** domain, i.e., let*

$$(-1)^{i-1}H[\gamma_i](x_1,x_2) \;\; \textit{be positive semi-definite for} \;\; (x_1,x_2)\in G_{12} \;\; (i=1,2), \tag{28}$$

$$(-1)^{i-1}H[\eta_i](x_1,x_3) \;\; \textit{be positive semi-definite for} \;\; (x_1,x_3)\in G_{13} \;\; (i=1,2), \tag{29}$$

$$(-1)^{i-1}H[\zeta_i](x_2,x_3) \;\; \textit{be positive semi-definite for} \;\; (x_2,x_3)\in G_{23} \;\; (i=1,2), \tag{30}$$

*and let*

$$p_{220}(x_3) \geq 0, \quad p_{202}(x_2) \geq 0, \tag{31}$$

$$p_{200}(\mathbf{x}) \leq 0, \quad p_{020}(x_3) \leq 0, \quad p_{002}(x_2) \leq 0, \tag{32}$$

$$p_{000}(x_2,x_3) \geq 0.$$

*Then problem* (25), (27) *is uniquely solvable.*
*Furthermore, if $E$ is **strongly convex**, i.e.,*

$$(-1)^{i-1}H[\gamma_i](x_1,x_2) \;\; \textit{is positive definite for} \;\; (x_1,x_2)\in G_{12} \;\; (i=1,2), \tag{33}$$

$$(-1)^{i-1}H[\eta_i](x_1,x_3) \;\; \textit{is positive definite for} \;\; (x_1,x_3)\in G_{13} \;\; (i=1,2), \tag{34}$$

$$(-1)^{i-1}H[\zeta_i](x_2,x_3) \;\; \textit{is positive definite for} \;\; (x_2,x_3)\in G_{23} \;\; (i=1,2), \tag{35}$$

*then the solution of problem* (25), (27) *belongs to* $C^{2,2,2}(\overline{E})$.

**Corollary 3.** *Let conditions* (28)–(32) *hold. Then problem* (26), (27) *has the Fredholm property. Furthermore, if conditions* (33)–(35) *hold, then every solution of problem* (26), (27) *belongs to* $C^{2,2,2}(\overline{E})$.

**Remark 2.** In a strongly convex domain the boundary conditions (2) are equivalent to the boundary conditions

$$u\big|_{\partial E} = \psi_1(\mathbf{x}), \quad u^{(2,0,0)}\big|_{\partial E} = \psi_2(\mathbf{x}), \quad u^{(2,2,0)}\big|_{\partial E} = \psi_3(\mathbf{x}).$$

**Remark 3.** Without the requirement that the domain $E$ be strongly convex the solution of problem (1), (2) may not belong to $C^{2,2,2}(\overline{E})$.

As an example, in the domain $E = \{(x_1, x_2, x_3) : \ x_1^4 + x_2^4 + x_3^4 < 1\}$ consider the problem

$$u^{(2,2,2)} = 0, \tag{36}$$

$$u\big|_{\partial E} = 0, \quad u^{(2,0,0)}\big|_{\partial E} = 2, \quad u^{(2,2,0)}\big|_{\partial E} = 0. \tag{37}$$

$E$ is a convex domain. However, $E$ is not strongly convex, since the Hessian matrices mentioned in Theorem 5 are positive semi-definite rather than positive definite along the three "main meridians"

$$\begin{cases} x_1^4 + x_2^4 = 1 \\ x_3 = 0 \end{cases}, \quad \begin{cases} x_1^4 + x_3^4 = 1 \\ x_2 = 0 \end{cases}, \quad \text{and} \quad \begin{cases} x_2^4 + x_3^4 = 1 \\ x_1 = 0 \end{cases}.$$

As a result, the unique solution of problem (36), (37) $u(\mathbf{x}) = x_1^2 - \sqrt{1 - x_2^4 - x_3^4}$ does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,1,0)}$ and $u^{(0,0,1)}$ are discontinuous along the third "main meridian".

It is worth noticing that problem (36), (37) considered in the unit ball $E = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < 1\}$ has a unique solution $u(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 1$ which belongs to $C^{2,2,2}(\overline{E})$. Such contrast is explained by the fact that the unit ball is a strongly convex domain.

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