On a Number of Zeros of Nontrivial Solutions to Second Order Singular Linear Differential Equations

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On a finite interval $]a,b[$, we consider the linear differential equation

$$u'' = p(t)u, \tag{1}$$

where $p :]a,b[\to \mathbb{R}$ is a measurable function, satisfying the condition

$$\int_a^b (t-a)(b-t)|p(t)|\,dt < +\infty. \tag{2}$$

We are mainly interested in the case where the function p has nonintegrable singularity at least at one of the boundary points of the interval $]a,b[$, i.e. the case, where

$$\int_a^b |p(t)|\,dt = +\infty.$$

A continuous function $u : [a,b] \to \mathbb{R}$ is said to be a solution to equation (1) if it is absolutely continuous together with u' on every closed interval contained in $]a,b[$ and satisfies equation (1) almost everywhere on $]a,b[$.

Following A. Wintner [5], we call equation (1) to be **disconjugate** on $[a,b]$ if its every nontrivial solution has no more than one zero on this interval.

In this report, we give unimprovable in a certain sense conditions under which equation (1) is disconjugate on $[a,b]$, or every its nontrivial solution has no more than two zeros on $[a,b]$. They are generalizations of the classical results by Lyapunov [4] and Hartman-Wintner [2] (see also [1], Ch. XI, § 5).

We use the following notations.

$$[x]_- = \frac{|x|-x}{2};$$

$C([a,b])$ and $L([a,b])$ are the spaces of continuous on $[a,b]$ and Lebesgue integrable on $[a,b]$ real functions, respectively;

$L_{loc}(]a,b[)$ is the space of real functions which are Lebesgue integrable on every closed interval contained in $]a,b[$;

$$I_1(p) = \int_a^{t_1} (t-a)[p(t)]_-\,dt, \quad I_2(p) = \int_a^{t_2} \frac{(t-a)(t_2-t)}{t_2-a}[p(t)]_-\,dt,$$

where the numbers $t_1 \in]a,b[$ and $t_2 \in]a,b[$ are chosen so that

$$\int_a^{t_1} (t-a)[p(t)]_-\,dt = \int_{t_1}^b (b-t)[p(t)]_-\,dt,$$

$$\int_a^{t_2} \frac{(t-a)(t_2-t)}{t_2-a}\,[p(t)]_-\,dt = \int_{t_2}^b \frac{(t-t_2)(b-t)}{b-t_2}\,[p(t)]_-\,dt. \tag{3}$$

If for some $t_0 \neq t_1$ the equality

$$\int_a^{t_0} (t-a)[p(t)]_-\,dt = \int_{t_0}^b (b-t)[p(t)]_-\,dt$$

is satisfied, then

$$\int_{t_0}^{t_1} (t-a)[p(t)]_-\,dt = 0.$$

Consequently, for every function $p \in L_{loc}(]a,b[)$, satisfying condition (2), the number $I_1(p)$ is defined uniquely.

If $[p(t)]_- \not\equiv 0$, then the number t_2 is defined uniquely from equality (3), and thus the number $I_2(p)$ is defined uniquely as well.

Moreover, if $[p(t)]_- \not\equiv 0$ and (2) holds, then

$$I_1(p) < \int_a^b \frac{(t-a)(b-t)}{b-a}\,[p(t)]_-\,dt. \tag{4}$$

If $[p(t)]_- \not\equiv 0$ and $p \in L([a,b])$, then

$$I_1(p) < \frac{b-a}{4} \int_a^b [p(t)]_-\,dt, \quad I_2(p) < \frac{b-a}{16} \int_a^b [p(t)]_-\,dt. \tag{5}$$

It has been proved by A. M. Lyapunov [4] that if $p \in C([a,b])$ and

$$\int_a^b [p(t)]_-\,dt \leq \frac{4}{b-a}\,, \tag{6}$$

then equation (1) is disconjugate. Hence it easily follows that if

$$\int_a^b [p(t)]_-\,dt \leq \frac{16}{b-a}\,,$$

then every nontrivial solution to equation (1) has no more than two zeros.

It has been shown by P. Hartman and A. Wintner [2] that equation (1) is disconjugate if $p \in C([a,b])$ and instead of (6) the more general condition

$$\int_a^b (t-a)(b-t)[p(t)]_-\,dt \leq b-a$$

is satisfied. This result is valid also for a singular case, when the function $p \in L_{loc}(]a,b[)$ satisfies condition (2) (see [3], Lemma 2.5).

We prove the following theorems.

Theorem 1. *If along with* (2) *the condition*

$$I_1(p) \le 1 \tag{7}$$

holds, then equation (1) *is disconjugate.*

Theorem 2. *If along with* (2) *the condition*

$$I_2(p) \le 1 \tag{8}$$

holds, then every nontrivial solution to equation (1) *has no more than two zeros on* $[a,b]$.

According to inequalities (4) and (5), Theorems 1 and 2 are generalizations of the above mentioned results by Lyapunov and Hartman-Wintner.

Remark 1. Inequality (7) in Theorem 1 (inequality (8) in Theorem 2) is unimprovable in the sense that it cannot be replaced by the inequality $I_1(p) < 1+\varepsilon$ (by the inequality $I_2(p) < 1+\varepsilon$) no matter how small $\varepsilon > 0$ would be.

Remark 2. For inequality (7) to be satisfied, it is sufficient that for some $t_0 \in]a,b[$ the inequalities

$$\int\limits_a^{t_0} (t-a)[p(t)]_- \, dt \le 1, \quad \int\limits_{t_0}^{b} (b-t)[p(t)]_- \, dt \le 1$$

hold. And if for some $t_0 \in]a,b[$ the inequalities

$$\int\limits_a^{t_0} (t-a)(t_0-t)[p(t)]_- \, dt \le t_0 - a, \quad \int\limits_a^{t_0} (t-t_0)(b-t)[p(t)]_- \, dt \le b - t_0$$

are satisfied, then inequality (8) is also satisfied.

Theorems 1 and 2 yield new and optimal in a certain sense conditions guaranteeing the unique solvability of the Dirichlet singular boundary value problem

$$u'' = p(t)u + q(t), \tag{9}$$
$$u(a) = c_1, \quad u(b) = c_2, \tag{10}$$

where p, $q \in L_{loc}(]a,b[)$, and $c_i \in \mathbb{R}$ $(i=1,2)$.

Theorem 3. *If along with* (2) *and* (7) *the condition*

$$\int\limits_a^b (t-a)(b-t)|q(t)| \, dt < +\infty \tag{11}$$

is satisfied, then problem $(9), (10)$ *has one and only one solution.*

Example 1. Let $\alpha < 2$,

$$\delta = (2-\alpha)\Big(\frac{2}{b-a}\Big)^{2-\alpha}, \quad p(t) = -\delta\bigg(\frac{b-a-|b+a-2t|}{2}\bigg)^{-\alpha} \ \text{ for } \ a < t < b,$$

and let q be the function satisfying condition (11). Then

$$\int\limits_a^b (t-a)(b-t)[p(t)]_- \, dt = \frac{4-\alpha}{3-\alpha}(b-a) > b-a,$$

i.e. the Lyapunov–Hartman–Wintner condition is violated. On the other hand,

$$I_1(p) = 1,$$

and by Theorem 3 problem (9), (10) is uniquely solvable.

Theorem 4. *Let conditions* (2)*,* (8)*, and* (11) *hold and there exist a function* $p_0 \in L_{loc}(]a,b[)$ *such that*

$$p(t) \le p_0(t) \le 0 \ \textit{ for } \ a < t < b, \quad \operatorname{mes}\big\{t \in]a,b[: \ p(t) < p_0(t)\big\} > 0,$$

and the boundary value problem

$$u'' = p_0(t)u; \ \ u(a) = 0, \ \ u(b) = 0$$

has a positive on the open interval $]a,b[$ *solution. Then problem* (9), (10) *has one and only one solution.*

Corollary 1. *Let*

$$p(t) < -\Big(\frac{\pi}{b-a}\Big)^2 \ \textit{ for } \ a < t < b,$$

and let conditions (2)*,* (8)*, and* (11) *be satisfied. Then problem* (9), (10) *has one and only one solution.*

Example 2. Let $\alpha < -2$,

$$0 < \delta \le \Big(1 - \frac{\pi^2}{24}\Big)(2-\alpha)\Big(\frac{2}{b-a}\Big)^{2-\alpha},$$
$$p(t) = -\Big(\frac{\pi}{b-a}\Big)^2 - \delta\bigg(\frac{b-a-|b+a-2t|}{2}\bigg)^{-\alpha} \ \text{ for } \ a < t < b,$$

and let q be the function satisfying condition (11). Then $I_2(p) \le 1$, and according to the above corollary problem (9), (10) has one and only one solution.

References

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