Dirichlet and Neumann Type Boundary Value Problems for Second Order Linear Differential Equations

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On a finite interval $[a,b]$, we consider the differential equation

$$u'' = p(t)u + q(t) \tag{1}$$

with one of the following two types boundary conditions:

$$u(a) = \ell_1 u'(a) + c_1, \quad u(b) = \ell_2 u'(b) + c_2; \tag{2}$$

$$u'(a) = \ell_1 u(a) + c_1, \quad u'(b) = \ell_2 u(b) + c_2, \tag{3}$$

where $p, q : [a,b] \to \mathbb{R}$ are Lebesgue integrable functions, and ℓ_i and c_i $(i = 1, 2)$ are real numbers.

If $\ell_1 = \ell_2 = 0$, then problems $(1), (2)$ and $(1), (3)$ are the Dirichlet and the Neumann problems, respectively, to the investigation of which a wide literature is devoted (see, e.g. [1–3, 6, 7] and the references therein). If $|\ell_1| + |\ell_2|$ is a sufficiently small positive number, then the above mentioned problems we naturally call Dirichlet and Neumann type problems.

In case, where $\ell_1 \geq 0$, $\ell_2 \leq 0$, the optimal in a certain sense sufficient conditions for the unique solvability of problem $(1), (3)$ are established in [4].

In the general case both problem $(1), (2)$ and problem $(1), (3)$ still remain little studied. The results given in this report fill to some extent the existing gap.

Below we use the following notation.

$$[x]_+ = \frac{|x| + x}{2}\,, \quad [x]_- = \frac{|x| - x}{2}\,, \quad \|p\| = \int_a^b |p(t)|\,dt.$$

Theorem 1. *If $\ell_1 \geq 0$, $\ell_2 \leq 0$, and*

$$\int_a^b (t - a + \ell_1)(b - t - \ell_2)[p(t)]_-\,dt \leq b - a + \ell_1 - \ell_2, \tag{4}$$

then problem $(1), (2)$ has one and only one solution.

Example 1. Introduce depending on a positive parameter γ functions

$$k_\gamma(x) = (\gamma + 3)x^\gamma - x^{2\gamma+2} \;\text{ for }\; 0 \leq x \leq 1, \quad k_\gamma(x) = k_\gamma(2 - x) \;\text{ for }\; 1 < x \leq 2,$$

$$v_\gamma(x) = x \exp\Big(-\frac{x^{\gamma+2}}{\gamma + 2}\Big) \;\text{ for }\; 0 \leq x \leq 1, \quad v_\gamma(x) = v_\gamma(2 - x) \;\text{ for }\; 1 < x \leq 2.$$

For an arbitrarily fixed $\varepsilon \in \,]0,1[\,$, set

$$\gamma = \frac{2}{\varepsilon}\,, \quad \ell_1 = (b-a)/(2-2^{-\gamma-1}) \;\text{ for }\; \ell_2 = -\ell_1,$$
$$p(t) = -(b-a)^{-2} k_\gamma\Big(\frac{2(t-a)+b-a}{2(b-a)}\Big) \;\text{ for }\; a \le t \le b.$$

Then

$$\int_a^b (t-a+\ell_1)(b-t-\ell_2)[p(t)]_-\,dt < (1+\varepsilon)(b-a+\ell_1-\ell_2). \tag{5}$$

On the other hand, the homogeneous problem

$$u'' = p(t)u, \tag{1_0}$$
$$u(a) = \ell_1 u'(a), \quad u(b) = \ell_2 u'(b) \tag{2_0}$$

has a nontrivial solution

$$u(t) \equiv v_\gamma\Big(\frac{2(t-a)+b-a}{2(b-a)}\Big).$$

Consequently, condition (4) is unimprovable in the sense that it cannot be replaced by condition (5) no matter how small $\varepsilon > 0$ is.

Theorem 2. *If the inequalities*

$$\ell_1 \ge 0, \;\; 0 < \ell_2 \le b-a+\ell_1, \;\; b-a+\ell_1-\ell_2+\|p\| > 0, \tag{6}$$
$$\int_a^b (t-a+\ell_1)[p(t)]_-\,dt \le 1, \quad \ell_2\int_a^b (t-a+\ell_1)[p(t)]_+\,dt \le b-a+\ell_1-\ell_2 \tag{7}$$

hold, then problem (1), (2) *has one and only one solution.*

Remark 1. Condition (6) cannot be replaced by the condition

$$\ell_1 \ge 0, \;\; 0 < \ell_2 \le b-a+\ell_1,$$

since if $\ell_2 = b-a+\ell_1$ and $\|p\| = 0$, then problem $(1_0), (2_0)$ has a nontrivial solution

$$u(t) \equiv \ell_1 + t - a.$$

Example 2. Let

$$\varepsilon \in \,]0,1[\,, \;\; \ell_1 = \ell_2 = \frac{b-a}{\varepsilon}\,, \;\; p(t) \equiv \Big(\frac{\varepsilon}{b-a}\Big)^2.$$

Then along with (6) the condition

$$\int_a^b (t-a+\ell_1)[p(t)]_-\,dt < 1, \quad \ell_2\int_a^b (t-a+\ell_1)[p(t)]_+\,dt < (1+\varepsilon)(b-a+\ell_1-\ell_2) \tag{8}$$

is satisfied. Nevertheless, problem $(1_0), (2_0)$ has a nontrivial solution

$$u(t) \equiv \exp\Big(\frac{\varepsilon(t-a)}{b-a}\Big).$$

Therefore, condition (7) cannot be replaced by condition (8) no matter how small $\varepsilon > 0$ is.

Theorem 3. *If the conditions*

$$\ell_2 \geq \ell_1 \geq 0, \ \ \ell_2 + \|p\| > 0, \tag{9}$$

$$\int_a^b (1 + \ell_1(t-a))(b-t)[p(t)]_- \, dt \leq 1 + \ell_1(b-a), \tag{10}$$

$$\ell_1 + \int_a^b (1 + \ell_2(t-a))[p(t)]_+ \, dt \leq \ell_2 \tag{11}$$

are satisfied, then problem $(1),(3)$ *has one and only one solution.*

Remark 2. The inequality $\ell_2 + \|p\| > 0$ cannot be omitted from condition (9). Indeed, if $\|p\| = 0$ and $\ell_2 = \ell_1 = 0$, then conditions $(10),(11)$ hold but nevertheless equation (1_0) has a nontrivial solution

$$u(t) \equiv 1,$$

satisfying the homogeneous boundary conditions

$$u'(a) = \ell_1 u(a), \quad u'(b) = \ell_2 u(b). \tag{3_0}$$

Example 3. Introduce the function

$$r(x) = \frac{\exp(x) - \exp(-x)}{x(\exp(x) + \exp(-x))} \ \text{ for } \ x \geq 0.$$

For an arbitrarily given $\varepsilon \in]0,1[$, we choose $\delta > 0$ such that

$$1 + \delta^2 r(\delta) < (1+\varepsilon) r(\delta).$$

Let

$$p(t) \equiv \Big(\frac{\delta}{b-a}\Big)^2, \ \ \ell_1 = 0, \ \ \ell_2 = \frac{r(\delta)}{b-a}\, \delta^2.$$

Then conditions (9), (10) hold, and instead of (11) the inequality

$$\ell_1 + \int_a^b (1 + \ell_2(t-a))[p(t)]_+ \, dt < (1+\varepsilon)\ell_2 \tag{12}$$

is satisfied. On the other hand, the homogeneous problem $(1_0),(3_0)$ has a nontrivial solution

$$u(t) \equiv \exp\Big(\frac{\delta(t-a)}{b-a}\Big) + \exp\Big(-\frac{\delta(t-a)}{b-a}\Big).$$

Consequently, condition (10) is unimprovable in the sense that it cannot be replaced by condition (12) no matter how small $\varepsilon > 0$ is.

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