Asgeirsson Principle and Exact Boundary Controllability Problems for One Class of Hyperbolic Systems

Sergo Kharibegashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia

E-mail: kharibegashvili@yahoo.com

In the domain $D_T: \ 0<x<l, \ 0<t<T$, of the plane O_{xt} of independent variables x, t consider a hyperbolic system of the following form

$$u_{tt}-Au_{xx}=F(x,t), \ \ (x,t)\in D_T, \tag{1}$$

where A is a symmetric positively defined constant square matrix of order n, $F=(F_1(x,t),\dots,F_n(x,t))$ is given and $u=(u_1(x,t),\dots,u_n(x,t))$ – unknown vector-functions, $n\geq 2$.

For system (1) consider an initial-boundary problem with the following statement: in the domain D_T find a solution $u=u(x,t)$ to system (1) that satisfies the following initial conditions

$$u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x), \ \ 0<x<l, \tag{2}$$

and the boundary conditions

$$u(0,t)=\mu_1(t), \quad u(l,t)=\mu_2(t), \ \ 0<t<T, \tag{3}$$

where

$$\varphi=(\varphi_1(x),\dots,\varphi_n(x)), \ \ \psi=(\psi_1(x),\dots,\psi_n(x)), \ \ \mu_i(t)=(\mu_{i1}(t),\dots,\mu_{in}(t)), \ \ i=1,2,$$

are given vector-functions.

As is known, problem (1), (2), (3) is posed correctly. We consider generalized solutions u of this problem in the space $C_0(\overline{D}_T)$ in the sense of the theory of distribution. Here the space $C_0(\overline{D}_T)$ is obtained by completion of the space $C^1(\overline{D}_T)$ with respect to the norm

$$\|u\|_{C_0(\overline{D}_T)}=\|u\|_{C(\overline{D}_T)}+\|u_t(x,0)\|_{C([0,l])},$$

and consists of continuous vector-functions u from $\overline{D}_T$ having continuous classical derivative u_t for $t=0$, $x\in[0,l]$. In this case, from the data of problem (1), (2), (3), i.e. from φ, ψ, μ_1, μ_2 and F, we require that

$$\varphi\in C([0,l]), \ \ \psi\in C([0,l]), \ \ \mu_i\in C_0([0,T]), \ \ i=1,2; \ \ F\in C(\overline{D}_T), \tag{4}$$

and at the points $O(0,0)$ and $O_1(0,0)$ there are valid the following necessary conditions of agreement

$$\mu_1(0)=\psi(0), \ \ \mu_2(0)=\varphi(l), \ \ \mu_1'(0)=\psi(0), \ \ \mu_2'(0)=\psi(l), \tag{5}$$

where the space $C_0([0,T])$ is obtained by completion of the space $C^1([0,T])$ with respect to the norm

$$\|\mu\|_{C_0([0,T])}=\|\mu\|_{C([0,T])}+\|\mu'(0)\|$$

and consists of the traces of vector-functions from the space $C_0(\overline{D}_T)$ on the side $\{x=0,\ 0\le t\le T\}$ of the rectangle D_T. At fulfillment of conditions (4), (5), problem (1), (2), (3) has a unique solution u in the space $C_0(\overline{D}_T)$. This solution will be a classical solution in the space $C^2(\overline{D}_T)$ if instead of (4) we require that

$$\varphi\in C^2([0,l]),\ \ \psi\in C^1([0,l]),\ \ \mu_i\in C^2([0,T]),\ \ i=1,2;\ \ F\in C^1(\overline{D}_T),$$

besides, in this case, at the points $O(0,0)$ and $O_1(0,0)$, together with (5) should be additionally fulfilled the following conditions of agreement

$$\mu_1''(0)-A\varphi''(0)=F(0,0),\ \ \mu_2''(0)-A\varphi''(l)=F(l,0).$$

Problem (1), (2), (3) is said to be controllable, if for "arbitrary" initial data φ, ψ and the right-hand side F of system (1), there exist appropriate "control" vector-functions μ_1 and μ_2 such that the solution of problem (1), (2), (3) satisfies the conditions

$$u(x,T)=u_t(x,T)=0,\ x\in[0,l]. \tag{6}$$

Denote by k_i the characteristic numbers of the matrix A, and by v_i – the corresponding eigenvectors, i.e. $Av_i=k_iv_i$. $i=1,\dots,n$. According to our requirements imposed on the matrix A we have

$$k_i=\lambda_i^2,\ \ \lambda_i=const>0,\ \ i=1,\dots,n. \tag{7}$$

Due to (7) the hyperbolic system (1) has the following families of characteristic lines

$$x+\lambda_it=const\ \text{ and }\ x-\lambda_it=const,\ \ i=1,\dots,n.$$

Denote by K a square matrix of order n whose columns are vectors $v_1,\dots,v_n$. It is obvious that $\det K\neq 0$ and denote by $w_1,\dots,w_n$ the components of the vector $K^{-1}u$ where u is a solution of system (1).

Denote by $P_0^iP_1^iP_2^iP_3^i$ the characteristic parallelogram whose sides $P_0^iP_1^i$ and $P_2^iP_3^i$ belong to the family of characteristic lines $x-\lambda_it=const$, while sides $P_0^iP_2^i$ and $P_1^iP_3^i$ belong to the characteristic lines $x+\lambda_it=const$, besides, the coordinate of point P_0^i with respect to the variable t exceeds the coordinates of the rest points P_1^i,P_2^i and P_3^i with respect to the same variable, $i=1,\dots,n$.

Generalized Asgeirsson principle: for the components $w_1,\dots,w_n$ of the vector $K^{-1}u$, where $u\in C_0(\overline{D}_T)$ is a generalized solution of system (1), the following equalities

$$w_i(P_0^i)=w_i(P_1^i)+w_i(P_2^i)-w_i(P_3^i)+\frac{1}{2\lambda_i}\int\limits_{P_0^iP_1^iP_2^iP_3^i}K^{-1}F(x,t)\,dx\,dt,\ \ i=1,\dots,n,$$

are valid, where $P_0^iP_1^iP_2^iP_3^i$ is an arbitrary characteristic parallelogram lying in $\overline{D}_T$.

Below, for simplicity of presentation we will assume that $F=0$.

Remark 1. If $T=\frac{l}{\lambda_i}$, $1\le i\le n$, then for existence of the solution $u=u(x,t)\in C_0(\overline{D}_T)$ of problem (1), (2), (3), satisfying condition (6) it is necessary that the data of this problem φ and ψ satisfy the following condition

$$\widetilde{\varphi}_i(0)+\widetilde{\varphi}_i(l)+\frac{1}{\lambda_i}\int\limits_0^l\widetilde{\psi}_i(\xi)\,d\xi=0, \tag{8}$$

where

$$\widetilde{\varphi}=(\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n)=K^{-1}\varphi,\ \ \widetilde{\psi}=(\widetilde{\psi}_1,\dots,\widetilde{\psi}_n)=K^{-1}\psi.$$

The proof of the following theorem is based on the generalized Asgeirsson principle.

Theorem. *Let* $T \geq T_0 = \max\limits_{1 \leq i \leq n} \frac{l}{\lambda_i}$ *and the vector-functions* $\varphi \in C([0,l])$, $\psi \in C([0,l])$ *be given which satisfy conditions* (8) *for* $i = 1, \dots, n$. *Then there exist vector-functions* $\mu_1, \mu_2 \in C_0([0,T])$ *satisfying the condition of agreement* (5) *such that the solution* $u \in C_0(\overline{D}_T)$ *of problem* (1), (2), (3) *satisfies condition* (6).

Remark 2. If $T < T_0 = \max\limits_{1 \leq i \leq n} \frac{l}{\lambda_i}$, then not for all $\varphi \in C([0,l])$, $\psi \in C([0,l])$ problem (1), (2), (3) is exactly controllable.

Remark 3. At fulfillment of the conditions of the above theorem, uniqueness of the vector-functions μ_1 and μ_2 will hold when $\lambda_0 := \lambda_1 = \lambda_2 = \cdots = \lambda_n$ for $T = T_0 = \frac{l}{\lambda_0}$ and violated when $T > \lambda_0$.