# On Exact Solutions of Karman's Equation in Nonlinear Theory of Gas Dynamics

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In the plane of independent variables $x$ and $y$ consider quasilinear Karman's equation, arising in a variety of physical problems such as nonlinear vibrations, and irrotational transonic flows of baritropic gas [1–4, 6, 12],

$$(u_x)^\alpha u_{xx} - u_{yy} = 0. \tag{1}$$

Equation (1) is considered in the class of hyperbolic solutions which in this case is determined by the condition

$$u_x > 0. \tag{2}$$

Let

$$m := \frac{\alpha}{2(\alpha+2)}\,, \quad -2 \neq \alpha \in \mathbb{R} := (-\infty, +\infty). \tag{3}$$

**Theorem.** *If the condition $m \in \mathbb{N} := \{1, 2, 3, \dots\}$ is fulfilled, then the general classical solution $u \in C^2$ of equation* (1) *is given by the formulas*

$$\begin{cases} x = (X-Y)^{2m+1}\,\dfrac{\partial^{2m}}{\partial X^m \partial Y^m}\,\dfrac{F(X)-G(Y)}{X-Y}\,, \\[2mm] y = m[2(1-2m)]^{2m}\,\dfrac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}}\,\dfrac{F'(X)-G'(Y)}{X-Y}\,, \\[2mm] u = m[2(1-2m)]^{2m}\bigg[\Big(\dfrac{m-1}{2m-1}\,X + \dfrac{m}{2m-1}\,Y\Big)\,\dfrac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}}\,\dfrac{F'(X)-G'(Y)}{X-Y} \\[2mm] \qquad\qquad - \dfrac{m-1}{2m-1}\,\dfrac{\partial^{2m-3}}{\partial X^{m-2}\partial Y^{m-1}}\,\dfrac{F'(X)-G'(Y)}{X-Y}\bigg] \ \ \textit{for} \ \ m = 2, 3, \dots \end{cases} \tag{4}$$

*and*

$$\begin{cases} x = -2[F(X)-G(Y)] + [F'(X)+G'(Y)](X-Y), \\[2mm] y = \dfrac{4[F'(X)-G'(Y)]}{X-Y}\,, \\[2mm] u = \dfrac{4[YF'(X)-XG'(Y)]}{X-Y} \ \ \textit{for} \ \ m = 1. \end{cases} \tag{5}$$

*Here $F, G \in C^{m+1}$ are arbitrary functions with respect to the variables $X$ and $Y$, respectively.*

***Proof.*** Let us introduce the Riemann invariants of equation (1) as independent variables

$$\begin{cases} X = q + \dfrac{2}{\alpha+2}\,p^{\frac{\alpha+2}{2}}, \\[2mm] Y = q - \dfrac{2}{\alpha+2}\,p^{\frac{\alpha+2}{2}}, \end{cases} \tag{6}$$

in terms of which equation (1) can be rewritten in the form of a system of equations of the first order [7,8]

$$\begin{cases} X_y - p^{\frac{\alpha}{2}} X_x = 0, \\ Y_y + p^{\frac{\alpha}{2}} Y_x = 0. \end{cases} \tag{7}$$

Here $p := u_x$, $q := u_y$.

In system (7), we choose $X$ and $Y$ as the independent variables, while $x(X,Y)$ and $y(X,Y)$ as the desired functions. Applying the formulas of differentiation of implicit functions of two variables

$$x_X = DY_y, \quad x_Y = -DX_y, \quad y_X = -DY_x, \quad y_Y = DX_x,$$

where $D := \frac{D(x,y)}{D(X,Y)}$ is the Jacobian of transformation, from system (7) we obtain

$$\begin{cases} x_X - p^{\frac{\alpha}{2}} y_X = 0, \\ x_Y + p^{\frac{\alpha}{2}} y_Y = 0. \end{cases} \tag{8}$$

Here $p^{\frac{\alpha}{2}} = \big\{\frac{1}{2(1-2m)}(X-Y)\big\}^{2m}$ due (2), (3) and (6).

Eliminating the function $y(X,Y)$ from system (8) we obtain that the function $x(X,Y)$ satisfies the Euler–Poisson–Darboux–Riemann equation [4,10]

$$x_{XY} + \frac{m}{X-Y}\, x_X - \frac{m}{X-Y}\, x_Y = 0. \tag{9}$$

By a similar way for the function $y(X,Y)$ we get

$$y_{XY} - \frac{m}{X-Y}\, y_X + \frac{m}{X-Y}\, y_Y = 0. \tag{10}$$

General solutions of equations (9) and (10) under the conditions of the theorem have the following form [9,11]

$$\begin{cases} x = (X-Y)^{2m+1} \dfrac{\partial^{2m}}{\partial X^m \partial Y^m} \dfrac{F_1(X) - G_1(Y)}{X-Y}, \\ y = \dfrac{\partial^{2m-2}}{\partial X^{m-1} \partial Y^{m-1}} \dfrac{F_2(X) - G_2(Y)}{X-Y}, \end{cases} \tag{11}$$

respectively. Here $F_1, G_1 \in C^{m+2}$ and $F_2, G_2 \in C^{m+1}$ are arbitrary functions.

Taking into account (11), satisfying system (8), we get

$$F_2(X) = m[2(1-2m)]^{2m} F_1'(X), \quad G_2(Y) = m[2(1-2m)]^{2m} G_1'(Y). \tag{12}$$

Further, to obtain the final form of the function $u$, due (3), (6) and (8) we have

$$\begin{aligned} du &= p\,dx + q\,dy \\ &= p(x_X\,dX + x_Y\,dY) + q(y_X\,dX + y_Y\,dY) = (q + p^{\frac{\alpha+2}{2}}) y_X\,dX + (q - p^{\frac{\alpha+2}{2}}) y_Y\,dY \\ &= \Big(\frac{m-1}{2m-1}\, X + \frac{m}{2m-1}\, Y\Big) y_X\,dX + \Big(\frac{m}{2m-1}\, X + \frac{m-1}{2m-1}\, Y\Big) y_Y\,dY, \end{aligned}$$

whence

$$U_X = \Big(\frac{m-1}{2m-1}\, X + \frac{m}{2m-1}\, Y\Big) y_X, \quad U_Y = \Big(\frac{m}{2m-1}\, X + \frac{m-1}{2m-1}\, Y\Big) y_Y. \tag{13}$$

By virtue of the first equality in (13), we obtain

$$\begin{aligned} U(X,Y) &= \frac{m-1}{2m-1}\int Xy_X\,dX + \frac{m}{2m-1}\,Yy + \varphi(Y) \\ &= \frac{m-1}{2m-1}\Big(Xy - \int y\,dX\Big) + \frac{m}{2m-1}\,Yy + \varphi(Y), \end{aligned} \tag{14}$$

where $\varphi$ is an arbitrary function.

According to the second equality from (13), for definition of the function $\varphi$, we get

$$\frac{m-1}{2m-1}\Big(Xy_Y - \int y_Y\,dX\Big) + \frac{m}{2m-1}\,(y+Yy_Y) + \varphi'(Y) = \Big(\frac{m}{2m-1}\,X + \frac{m-1}{2m-1}\,Y\Big)y_Y. \tag{15}$$

By virtue of (10), we obtain

$$\int y_Y\,dX = \int\Big(\frac{Y-X}{m}\,y_{XY} + y_X\Big)\,dX = \frac{Y-X}{m}\,y_Y + \frac{1}{m}\int y_Y\,dX + y.$$

Thus, we have

$$\int y_Y\,dX = \frac{Y-X}{m-1}\,y_Y + \frac{m}{m-1}\,y \ \text{ for } \ m\neq 1.$$

Taking into account the latter equality, from (15) we obtain

$$\varphi'(Y)\equiv 0 \implies \varphi = const \ \text{ for } \ m=2,3,\dots . \tag{16}$$

Analogously, from (14) for $m=1$, we get

$$U(X,Y) = Yy + \varphi(Y). \tag{17}$$

According to the second equality from (13) for $m=1$, for definition of the function $\varphi$, we get

$$\varphi'(Y) = (X-Y)y_Y - y = -G_2'(Y) \implies \varphi(Y) = -G_2(Y). \tag{18}$$

Now, introducing the notation $F := F_1$, $G := G_1$ and taking into account (11), (12), (14), (16)–(18), we obtain (4) and (5), respectively. $\square$

**Remark.** In the case $m=1$, i.e. for $\alpha=-4$, the solution (5) of equation (1) by the method of Lee's group has been obtained in [5].

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