Structure of Nonoscillatory Solutions of Second Order Half-Linear Differential Equations

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We consider second order half-linear differential equations of the form

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{A}$$

where $\alpha > 0$ is a constant, $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$, $a \geqq 0$, and $\varphi_\alpha(u)$ is an odd function on $\mathbb{R}$ defined by

$$\varphi_\alpha(u) = |u|^{\alpha-1}u = |u|^\alpha \operatorname{sgn} u, \quad u \in \mathbb{R}.$$

It is known that all proper solutions of (A) are oscillatory, or else nonoscillatory. Equation (A) itself is said to be oscillatory (or nonoscillatory) if all of its proper solutions are oscillatory (or nonoscillatory). We are concerned exclusively with the nonoscillatory equation (A) with $p(t)$ and $q(t)$ satisfying the conditions

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt = \infty \ \text{ and } \ \int_a^\infty q(t)\,dt < \infty. \tag{1}$$

Extensive use is made of the functions $P_\alpha(t)$ and $\rho(t)$ defined by

$$P_\alpha(t) = \int_a^t p(s)^{-\frac{1}{\alpha}}\,ds, \quad \rho(t) = \int_t^\infty q(s)\,ds.$$

The purpose of this paper is to report to the QUALITDE – 2020 a result that has recently been obtained in our efforts to gain precise information about the overall structure of solutions of nonoscillatory equations of the form (A).

We begin by noting that a simple criterion for nonoscillation of (A) is given by

$$P_\alpha(t)\rho(t)^{\frac{1}{\alpha}} \leqq \frac{\alpha}{1+\alpha} \ \text{ for all large } t.$$

Putting $D_\alpha x(t) = p(t)\varphi_\alpha(x'(t))$, we call it the quasi-derivative of $x(t)$. Let $x(t)$ be a nonocillatory solution of (A). Because of (1) both $x(t)$ and $D_\alpha x(t)$ are of the same sign and have the limits $x(\infty) = \lim\limits_{t\to\infty} x(t)$ and $D_\alpha x(\infty) = \lim\limits_{t\to\infty} D_\alpha x(t)$ in the extended real number system. There are three patterns of the pair $\{x(\infty), D_\alpha x(\infty)\}$, namely,

(i) $|x(\infty)| = \infty$, $0 < |D_\alpha x(\infty)| < \infty$;

(ii) $|x(\infty)| = \infty$, $D_\alpha x(\infty) = 0$;

(iii) $0 < |x(\infty)| < \infty$, $D_\alpha x(\infty) = 0$.

A solution of (A) satisfying (i), (ii) or (iii) are named, respectively, a *maximal*, *minimal*, or *intermediate solution*. If $x(t)$ is a nonoscillatory solution of (A), then the functions $u(t)$, $v(t)$ defined by

$$u(t) = \frac{D_\alpha x(t)}{\varphi_\alpha(x(t))}, \quad v(t) = -\frac{x(t)}{\varphi_{\frac{1}{\alpha}}(D_\alpha x(t))}$$

satisfy the first order differential equations

$$u' = -q(t) - \alpha p(t)^{-\frac{1}{\alpha}} |u|^{1+\frac{1}{\alpha}}, \tag{R1}$$

$$v' = -p(t)^{-\frac{1}{\alpha}} - \frac{1}{\alpha} q(t)|v|^{1+\alpha}. \tag{R2}$$

Conversely, it is shown that if (R1) or (R2) has a global solution, then (A) possesses a nonoscillatory solution. Our study was motivated by the ambitious conjecture that all nonoscillatory solutions can be reproduced from appropriate global solutions of (R1) or (R2). Equations (R1) and (R2) are referred to as the generalized Riccati differential equations (Riccati equations for short) associated with equation (A).

It turns out that the existence of these three types of solutions of (A) essentially depends on the convergence or divergence of the integrals

$$I = \int_a^\infty p(t)^{-\frac{1}{\alpha}} \rho(t)^{\frac{1}{\alpha}}\, dt \ \text{ and } \ J = \int_a^\infty q(t) P_\alpha(t)^\alpha\, dt. \tag{2}$$

Let us distinguish the following four cases:

$$\text{(i)} \ \ I < \infty \land J < \infty; \ \ \text{(ii)} \ \ I = \infty \land J < \infty; \ \ \text{(iii)} \ \ I < \infty \land J = \infty; \ \ \text{(iv)} \ \ I = \infty \land J = \infty. \tag{3}$$

Notice that (ii) holds only if $\alpha > 1$ and that (iii) holds only if $\alpha < 1$.

Analysis of the first three cases in (3) can be made without difficulty and leads to the following expected result.

Theorem 1. *If* (3)-(i) *holds, then* (A) *possesses a maximal solution and a minimal solution.*

Theorem 2. *If* (3)-(ii) *holds, then* (A) *possesses a maximal solution and an intermediate solution.*

Theorem 3. *If* (3)-(iii) *holds, then* (A) *possesses a minimal solution and an intermediate solution.*

What is anticipated for the case (3)-(iv) is the existence of at least one intermediate solution of (A). This, however, seems to be difficult to prove, and for now we have to be content with giving a less general result on the basis of the inequality

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}} \rho(t)^{1+\frac{1}{\alpha}}\, dt < \infty,$$

which is a necessary condition for nonoscillation of (A).

Theorem 4. *Let* (3)-(iv) *hold.* (A) *possesses an intermediate solution if*

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}}\rho(t)^{1+\frac{1}{\alpha}}\,ds \leqq (1+\alpha)^{-1-\frac{1}{\alpha}}\rho(t) \ \textit{ for all large } t. \tag{4}$$

Taking into account the duality between the integrals I and J in (2), one can formulate the following theorem which is also true.

Theorem 5. *Let* (3)-(iv) *hold.* (A) *possesses an intermediate solution if*

$$\int_a^t q(s)P_\alpha(s)^{1+\alpha}\,ds \leqq \Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}P_\alpha(t) \ \textit{ for all large } t.$$

Example. Let (A_0) denote a special case of (A) with $q(t)$ given by

$$q(t) = \Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}p(t)^{-\frac{1}{\alpha}}P_\alpha(t)^{-1-\alpha}.$$

It is clear that $p(t)$ and $q(t)$ satisfy (3)-(iv) and both Theorems 4 and 5 are applicable to (A_0). Notice that (A_0) has an exact intermediate solution $x_0(t) = P_\alpha(t)^{\frac{\alpha}{1+\alpha}}$.

The feature of our work is that all the solutions of (A) mentioned in the above theorems are reproduced from appropriate global solutions of the Riccati equations (R1) and (R2) whose existence is established by means of fixed point techniques. Such a systematic attempt at reproduction of nonoscillatory solutions of (A) from global solutions of the associated Riccati equations was undertaken for the first time by the present authors [2]. The merit of our approach is that the solutions sought can be represented as explicit exponential–integral formulas in terms of global solutions of (R1) or (R2).

It should be emphasized that some of the results presented here are already known (see, e.g., [1]), but our purpose is to show that an entirely different approach can be used to develop a systematic existence theory of nonoscillatory solutions for second order half-linear differential equations.

Remark. Needless to say, entirely parallel results can also be obtained for the nonoscillatory equation (A) with $p(t)$ and $q(t)$ satisfying

$$\int_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt < \infty \ \text{ and } \int_a^\infty q(t)\,dt = \infty. \tag{5}$$

Our article devoted to the study of two types of nonoscillation for equation (A) satisfying (1) and (5) combined will be published in the near future.

References

[1] Z. Došlá and I. Vrkoč, On an extension of the Fubini theorem and its applications in ODEs. *Nonlinear Anal.* **57** (2004), no. 4, 531–548.

[2] J. Jaroš, T. Kusano and T. Tanigawa, Nonoscillatory solutions of planar half-linear differential systems: a Riccati equation approach. *Electron. J. Qual. Theory Differ. Equ.* **2018**, Paper No. 92, 28 pp.