Perron's Effect of Value Change of Characteristic Exponents with a Countable Number of Uniformly Bounded Suslin's Sets

N. A. Izobov

Department of Differential Equations, Institute of Mathematics,
National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: izobov@im.bas-net.by

A. V. Il'in

Moscow State University, Moscow, Russia
E-mail: iline@cs.msu.su

Consider the linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^2, \ \ t \geq t_0, \tag{1}$$

with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_1(A) \leq \lambda_2(A) < 0$, being linear approximations for the nonlinear systems

$$\dot{y} = A(t)y + f(t,y), \ \ y \in \mathbb{R}^2, \ \ t \geq t_0, \tag{2}$$

likewise with infinitely differentiable so-called m-perturbations $f(t,y)$ of order $m > 1$ of smallness in the neighbourhood of the origin and possible growth outside of it:

$$\|f(t,y)\| \leq C_f \|y\|^m, \ \ y \in \mathbb{R}^2, \ \ t \geq t_0. \tag{3}$$

The known Perron's effect [7], [6, p. 50-51] of value change of characteristic exponents states the existence of systems (1) and (2) with 2-perturbation (3) such that all nontrivial solutions of system (2) turn out to be infinitely continuable and their characteristic exponents take only two values: one is negative, coinciding with the higher exponent $\lambda_2(A) < 0$ of the system of linear approximation (1) and the other one is positive (calculated incidentally in [3, p. 13-15]). Considering this effect as not full (not all nontrivial solutions of the perturbed system (2) take positive exponents), a great number of works were devoted to the investigation of its full version (all nontrivial solutions of system (2) are infinitely continuable to the right and have finite positive exponents (see our last works [4, 5]).

In particular, in these works we have obtained the above-mentioned full Perron's effect corresponding to various types of the collection $\Lambda(A,f) \subset (0,+\infty)$ of Lyapunov's characteristic exponents of all nontrivial solutions of the nonlinear system (2) with m-perturbation (3) for any fixed $m > 1$. In our last works, these bounded collections $\Lambda(A,f) \subset (0,+\infty)$ are completely described by Suslin's sets.

Noteworthy are the results, not connected with Perron's effect: for the exponentially stable systems (2) with perturbations (3) the collections $\Lambda_0(A,f) \subset (-\infty,0)$ of characteristic exponents of all their nontrivial solutions, emanating from a sufficiently small neighbourhood of the origin, of positive measure are realized in [2], whereas in [1] they are completely described by Suslin's sets.

In investigating the above (not full) Perron's effect, when all nontrivial solutions $y(t,c), y(t_0,c) = c \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ are infinitely continuable and have finite positive as well as negative Lyapunov's exponents $\lambda[y(\,\cdot\,,c)]$, forming respectively non-empty (one-element) sets $\Lambda_+(A,f)$ and $\Lambda_-(A,f)$ and all their collection $\Lambda(A,f)=\Lambda_+(A,f)\cup\Lambda_-(A,f)$, there arises, in particular, the question to what extent they may be common simultaneously (for one system (2)). The answer, as a consequence of a more general result, is obtained in the present report.

Here, for a countable number of uniformly bounded arbitrary Suslin's sets S_k, $k\in\mathbb{N}$ and a partitioning of a plane of initial values of solutions into the same number of domains and segments Π_k we have constructed systems (1) and (2) with m-perturbation (3) such that the characteristic exponents $\lambda[y(\,\cdot\,,c)]$ of nontrivial solutions of system (2) with the initial values $c\in\Pi_k$ make up the sets S_k, and the whole collection of exponents $\Lambda(A,f)$ of nontrivial solutions of that system is the union $\bigcup\limits_{k\in\mathbb{N}} S_k$. The consequence of that common result is the realization of the cases for: a finite number of arbitrary bounded Suslin's sets; two arbitrary, likewise bounded, Suslin's sets

$$S_+\subset(0,+\infty),\quad S_-\subset(-\infty,0),$$

forming the collection $\Lambda(A,f)=S_+\cup S_-$ of characteristic exponents of some system (2).

The following theorem is valid.

Theorem 1. *For any parameters $m>1$, $\lambda_1\le\lambda_2<0$ and any two sequences S_{in}, $n\in\mathbb{N}$, $i=1,2$, uniformly bounded by Suslin's sets*

$$S_{1n}\subset\big[\lambda_1+\varepsilon,b_1\big],\quad S_{2n}\subset\big[\max\{\lambda_2+\varepsilon,b_1\},b_2\big],\ \ n\in\mathbb{N},$$

with number $\varepsilon>0$, which are the sets of values respectively of the functions $\beta_{1n}(\,\cdot\,)$ and $\beta_{2n}(\,\cdot\,)$ of the 1st Baire class on every of the half-intervals $(n-1,n]$ and $[-n,-n+1)$, there exist:

1) *a system of linear approximation (1) with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_i(A)=\lambda_i$, $i=1,2$;*

2) *an infinitely differentiable m-perturbation $f(t,y)$ such that all nontrivial solutions $y(t,c)$ with the initial conditions*
$$y(t_0,c)=(c_1,c_2)\in\mathbb{R}^2\setminus\{\mathbf{0}\}$$
of the perturbed system (2) are infinitely continuable to the right, and their characteristic exponents $\lambda[y(\,\cdot\,,c)]$ form for every $n\in\mathbb{N}$ the collections
$$\Big\{\lambda\big[y(\,\cdot\,,(c_1,0))\big]:\ |c_1|\in(n-1,n]\Big\}=S_{1n},$$
$$\Big\{\lambda\big[y(\,\cdot\,,c)\big]:\ |c_2|\in(n-1,n]\Big\}=S_{2n},\quad \Lambda(A,f)=\bigcup_{i,n}S_{in},$$
on every of the above-mentioned intervals separately.

The theorem below gives us the answer to the question posed at the beginning of our report.

Theorem 2. *For any parameters $m>1$, $\lambda_1\le\lambda_2<0$ and arbitrary bounded Suslin's sets $S_-\subset(-\infty,0)$ and $S_+\subset(0,+\infty)$ there exist the nonlinear system (2) with the linear approximation (1), having characteristic exponents $\lambda_i(A)=\lambda_i$, $i=1,2$, and m-perturbation (3) such that all nontrivial solutions $y(t,c)$ are infinitely continuable and their Lyapunov's exponents $\lambda[y(\,\cdot\,,c)]$ form the sets*

$$\big\{\lambda[y(\,\cdot\,,c)]:\ c=(c_1,0)\neq0\big\}=S_-,\quad \big\{\lambda[y(\,\cdot\,,c)]:\ c_2\neq0\big\}=S_+.$$

This theorem is a direct consequence of Theorem 1 and its proof.

Acknowledgements

The present work is carried out under the financial support of Belarusian republican (project # Ф20Р-005) and Russian (project # 20-57-00001Bel) funds of fundamental researches.

References

[1] E. A. Barabanov and I. A. Volkov, The structure of the set of Lyapunov characteristic exponents of exponentially stable quasi-linear systems. (Russian) *Differentsial'nye Uravneniya* **30** (1994), no. 1, 3–19; translation in *Differential Equations* **30** (1994), no. 1, 1–15.

[2] N. A. Izobov, The number of characteristic and lower exponents of an exponentially stable system with higher-order perturbations. (Russian) *Differentsial'nye Uravneniya* **24** (1988), no. 5, 784–795, 916–917; translation in *Differential Equations* **24** (1988), no. 5, 510–519.

[3] N. A. Izobov, *Lyapunov Exponents and Stability.* Cambridge Scientific Publishers, 2012.

[4] N. A. Izobov and A. V. Il'in, On the Baire classification of positive characteristic exponents in the Perron effect of change of their values. (Russian) *Differ. Uravn.* **54** (2018), no. 11, 1435–1439; translation in *Differ. Equ.* **54** (2018), no. 11, 1409–1413.

[5] N. A. Izobov and A. V. Il'in, Construction of an arbitrary Suslin set of positive characteristic exponents in the Perron effect. (Russian) *Differ. Uravn.* **55** (2019), no. 4, 464–472; translation in *Differ. Equ.* **55** (2019), no. 4, 449–457.

[6] G. A. Leonov, *Chaotic Dynamics and Classical Theory of Motion Stability.* (Russian) NITs RKhD, Izhevsk, Moscow, 2006.

[7] O. Perron, Die Stabilitätsfrage bei Differentialgleichungen. (German) *Math. Z.* **32** (1930), no. 1, 703–728.