Dulac–Cherkas Functions for Van Der Pol Equivalent Systems

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The differential equation
$$\frac{d^2x}{dt^2}+\mu(x^2-1)\,\frac{dx}{dt}+x=0 \tag{1}$$
depending on the real parameter μ has been introduced by the Dutch engineer and physicist Balthasar van der Pol [4] in 1926 to describe self-oscillations in a triod circuit. If we replace t by $-t$ and μ by $-\mu$, then equation (1) remains invariant. Thus, to study the phase portrait of equation (1) we can restrict ourselves to the case $\mu\geq 0$. It is well-known (see, e.g., [3]) that (1) has for $\mu>0$ a unique limit cycle $\Gamma(\mu)$ which is orbitally stable and hyperbolic. For small μ, the periodic solutions $x=p(t,\mu)$ describing the limit cycle $\Gamma(\mu)$ behave like the solution of the harmonic oscillator, for large μ, $p(t,\mu)$ represents a relaxation oscillation. In what follows we derive differential systems which distinguish in their structure but whose phase portraits are topologically equivalent to that of the van der Pol equation (1). The reason to do this consists in the intension to find the most suitable form for studying the localization and the shape of the limit cycle for arbitrary values of the parameter.

The main tool for our investigation is the method of Dulac–Cherkas functions which was introduced by L. A. Cherkas in 1997 [1] as a generalization of Dulac method [2]. We recall the definition of Dulac–Cherkas function for the planar autonomous differential system
$$\frac{dx}{dt}=P(x,y),\quad \frac{dy}{dt}=Q(x,y) \tag{2}$$
in some open region $\mathcal{G}\subset\mathbb{R}^2$, where $P,Q\in C^1(\mathcal{G},\mathbb{R})$ and X is the vector field defined by (2).

Definition 1. A function $\Psi\in C^1(\mathcal{G},\mathbb{R})$ is called the Dulac–Cherkas function of system (2) in $\mathcal{G}$ if there exists a real number $\kappa\neq 0$ such that
$$\Phi(x,y,\kappa):=(\operatorname{grad}\Psi,X)+\kappa\Psi\operatorname{div}X>0\ \ (<0)\ \text{ in }\mathcal{G}. \tag{3}$$

In case $\kappa=1$, Ψ is a Dulac function.

Remark 1. Condition (3) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set of measure zero, and that no oval of this set is a limit cycle of (2).

For the sequel we introduce the subset $\mathcal{W}$ of $\mathcal{G}$ defined by $\mathcal{W}:=\{(x,y)\in\mathcal{G}:\ \Psi(x,y)=0\}$.

The following theorem can be found in [1,2].

Theorem 1. *Let Ψ be a Dulac–Cherkas function of* (2) *in $\mathcal{G}$. Then any limit cycle Γ of* (2) *located entirely in $\mathcal{G}$ has the following properties:*

(i) Γ *does not intersect $\mathcal{W}$;*

(ii) Γ *is hyperbolic;*

(iii) *the stability of Γ is determined by the sign of the expression $\kappa\Phi\Psi$ on Γ.*

Property (ii) has the strong consequence that the existence of a Dulac–Cherkas function implies that system (2) has no multiple limit cycle.

Theorem 2. *Let Ψ be a Dulac–Cherkas function of* (2) *in $\mathcal{G}$ such that the set $\mathcal{W}$ contains some oval $\mathcal{W}_0$ with the property that the open region $\mathcal{G}_0$ bounded by $\mathcal{W}_0$ belongs to $\mathcal{G}$ and that $\mathcal{G}_0 \cap \mathcal{W}$ is empty. Then there is no limit cycle in $\mathcal{G}_0$.*

Corollary 1. *Under the assumptions of Theorem 2, $\mathcal{W}_0$ can be used as interior boundary of a possible Poincaré–Bendixson annulus.*

The following result is also known [2].

Theorem 3. *Let $\mathcal{G}$ be a simply connected region where Ψ is a Dulac–Cherkas function of* (2) *such that $\mathcal{W}$ consists of one oval in $\mathcal{G}$. Then system* (2) *has at most one limit cycle in $\mathcal{G}$.*

Now we note that (1) can be rewritten as the system

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= x - \mu(x^2-1)y \end{aligned} \tag{4}$$

of Liénard type. The goal of our investigation is to construct Dulac–Cherkas functions for systems equivalent to the van der Pol system (4) such that the zero-set of these functions consists of a unique oval which can be used by Corollary 1 as interior boundary of a Poincaré–Bendixson annulus containing the unique limit cycle of the corresponding system. At the same time we study the problem whether a Dulac–Cherkas function for an equivalent system can be obtained by applying the equivalence relation to the known Dulac–Cherkas function. We start with the original van der Pol system.

Lemma 1. *The functions*

$$\Psi_a(x,y) := x^2 - 1 + y^2 \tag{5}$$

and

$$\Psi_b(x,y,\mu) := x^2 - \frac{8}{3} + \mu\Big(x - \frac{x^3}{3}\Big)y + y^2 \tag{6}$$

are Dulac–Cherkas functions for system (4) *in $\mathbb{R}^2$ for $\mu > 0$.*

The corresponding expressions (3) read

$$\begin{aligned} \Phi_a(x,-2,\mu) &= 2\mu(x^2-1)^2 \geq 0, \\ \Phi_b(x,-1,\mu) &= \frac{2}{3}\,\mu(x^2-2)^2 \geq 0 \end{aligned}$$

for $\kappa = -2$ and $\kappa = -1$, accordingly.

Remark 2. The zero-sets $\mathcal{W}_a$ and $\mathcal{W}_b(\mu)$ of the functions $\Psi_a(x,y)$ and $\Psi_b(x,y,\mu)$ consist of a unique oval for $\mu > 0$. Thus, these ovals can be used as interior boundaries for a Poincaré-Bendixson annulus. We note that the parameter dependent oval $\mathcal{W}_b(\mu)$ represent for small μ a better approximation of the van der Pol limit cycle $\Gamma(\mu)$.

For the first time the function $\Psi_a(x,y)$ was constructed by L. A. Cherkas in the paper [1]. Next we consider the system

$$\begin{aligned} \frac{d\overline{x}}{dt} &= -\overline{y}, \\ \frac{d\overline{y}}{dt} &= \overline{x} + (\mu - \overline{x}^2)\overline{y}, \end{aligned} \tag{7}$$

which we obtain from system (4) by the scaling $\overline{x} = \sqrt{\mu}\,x, \overline{y} = \sqrt{\mu}\,y$. The representations (4) and (7) are especially useful for small μ: if μ crosses the value 0, the unique limit cycle $\Gamma(\mu)$ in system (4) bifurcates from the family of circles with center at the origin, while in system (7) the limit cycle $\Gamma(\mu)$ bifurcates from the origin (Hopf bifurcation). We note that in the case $\mu = 0$ the phase portraits of these systems are not topologically equivalent.

Lemma 2. *The functions*

$$\overline{\Psi}_a(\overline{x},\overline{y},\mu) := \overline{x}^2 + \overline{y}^2 - \mu \tag{8}$$

and

$$\overline{\Psi}_b(\overline{x},\overline{y},\mu) := \overline{x}^2 + \overline{y}^2 - \frac{8}{3}\,\mu + \Big(\mu\overline{x} - \frac{\overline{x}^3}{3}\Big)\overline{y} \tag{9}$$

are Dulac–Cherkas functions for system (7) *in* $\mathbb{R}^2$ *for* $\mu > 0$*.*

Both ovals corresponding to the functions (8) and (9) can be used as interior boundaries for a Poincaré-Bendixson annulus. Now we study the singularly perturbed system

$$\begin{aligned} \frac{dx}{d\tau} &= -y, \\ \varepsilon\,\frac{dy}{d\tau} &= x - (x^2 - 1)y, \end{aligned} \tag{10}$$

which we get from system (4) by the scaling $t = \mu\tau$ and using the notation $\varepsilon = 1/\mu^2$. In the case $\mu = 1$ both system coincide such that the functions Ψ_a and Ψ_b defined in (5) and (6) are also Dulac–Cherkas functions of system (7). For $\mu \neq 1$, this scaling changes not only the velocity running along the trajectories but also the vector field such that $\overline{\Psi}_a$ and $\overline{\Psi}_b$ are not longer Dulac–Cherkas functions of system (7).

Lemma 3. *The functions*

$$\Psi_a(x,y,\varepsilon) := x^2 - 1 + \varepsilon y^2$$

and

$$\Psi_b(x,y,\varepsilon) := x^2 - \frac{8}{3} + \Big(x - \frac{x^3}{3}\Big)y + \varepsilon y^2$$

are Dulac–Cherkas functions for system (10) *in* $\mathbb{R}^2$ *for* $\varepsilon > 0$*.*

In the similar way we derive the following results for three other van der Pol equivalent systems.

Lemma 4. *The function*

$$\Psi(\xi,\eta,\mu) := \xi^2 - \frac{8}{3} + \mu\Big(\eta - \frac{\eta^3}{3}\Big)\xi + \eta^2$$

is a Dulac–Cherkas function for system

$$\begin{aligned}\frac{d\xi}{dt}&=-\eta,\\ \frac{d\eta}{dt}&=\mu\Big(\eta+\frac{\eta^3}{3}\Big)+\xi\end{aligned}$$

in $\mathbb{R}^2$ *for* $\mu>0$.

Lemma 5. *The function*

$$\Psi(\overline{\xi},\overline{\eta},\mu):=\overline{\xi}^2-\frac{8}{3}\,\mu+\Big(\mu\overline{\eta}-\frac{\overline{\eta}^3}{3}\Big)\overline{\xi}$$

is a Dulac–Cherkas function for system

$$\begin{aligned}\frac{d\overline{\xi}}{dt}&=-\overline{\eta},\\ \frac{d\overline{\eta}}{dt}&=\overline{\xi}+\mu\overline{\eta}-\frac{\overline{\eta}^3}{3}\end{aligned}$$

in $\mathbb{R}^2$ *for* $\mu>0$.

Lemma 6. *The function*

$$\Psi(\xi,\eta,\varepsilon):=\xi^2+\varepsilon\eta^2-\frac{8}{3}\,\varepsilon+\Big(\eta-\frac{\eta^3}{3}\Big)\xi$$

is a Dulac–Cherkas function for system

$$\begin{aligned}\frac{d\xi}{d\tau}&=-\eta,\\ \varepsilon\,\frac{d\eta}{d\tau}&=\xi+\eta-\frac{\eta^3}{3}\end{aligned}$$

in $\mathbb{R}^2$ *for* $\varepsilon>0$.

Finally for the van der Pol system we present an approach for the construction of an outer boundary for the Poincaré-Bendixson annulus which does not require an approximation of any orbit.

Theorem 4. *The algebraic ovales*

$$x^2+y^2=1$$

and

$$y^2+\mu yx\Big(2-\frac{x^2}{3}\Big)+(1+\mu^2)x^2-\frac{7\mu^2}{12}\,x^4+\frac{\mu^2}{18}\,x^6-C(\mu)=0$$

form a global algebraic Poincaré-Bendixson annulus for system (4).

In the proof of this theorem we describe a way how the function $C(\mu)$ depending on the parameter μ can be selected. Our approach implies the uniqueness of a limit cycle in the constructed Poincaré-Bendixson annulus.

References

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