Approximate Optimal Control on Semi-Axis for the Reaction-Diffusion Process with Fast-Oscillating Coefficients

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1 Introduction and setting of the problem

It is known that the local characteristics of processes in micro-inhomogeneous medium contain functions of the form $a(\frac{x}{\varepsilon})$, where $\varepsilon > 0$ is a small parameter. Passing to averaged parameters is an effective tool for studying such processes [10]. Such a procedure for parabolic operators was justified in [7]. Optimal control problems for parabolic equations with fast oscillating functions in the coefficients were investigated in [1,6,12,14]. General questions of the solvability of systems of the reaction-diffusion type were investigated in [4,9,13]. In this paper, we consider the optimal control problem on semi-axis for the reaction - diffusion equation with a coercive objective functional, whose coefficients contain fast oscillating functions.

More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\varepsilon \in (0,1)$ be a small parameter. In the cylinder $Q = (0,\infty) \times \Omega$, the controlled process is described by the evolutionary system

$$\begin{cases} \dfrac{\partial y}{\partial t} = \operatorname{div}\Big[a\Big(\dfrac{x}{\varepsilon}\Big)\nabla y\Big] - b\Big(\dfrac{x}{\varepsilon}\Big)f(y) + u(t,x), \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{t=0} = y_0^{\varepsilon}(x), \end{cases} \tag{1.1}$$

$$u \in U \subseteq L^2(Q), \tag{1.2}$$

$$J(y,u) = \int\limits_Q q_\varepsilon(t,x,y(t,x))y(t,x)\,dt\,dx + \gamma \int\limits_Q u^2(t,x)\,dt\,dx \longrightarrow \inf, \;\; \gamma > 0. \tag{1.3}$$

Under natural assumptions on parameters we prove the following limit equality

$$J(\overline{y}^{\,\varepsilon}, \overline{u}^{\,\varepsilon}) \longrightarrow J(\overline{y}, \overline{u}), \;\; \varepsilon \to 0,$$

where $\{\overline{y}^{\,\varepsilon}, \overline{u}^{\,\varepsilon}\}$ and $\{\overline{y}, \overline{u}\}$ are optimal processes of the perturbed problem (1.1)–(1.3) and the corresponding averaged problem.

2 Main results

We will consider the optimal control problem (1.1)–(1.3) under following assumptions:

a is a measurable, periodic, symmetric matrix satisfying the condition of uniform ellipticity and boundedness

$$\forall\, x \in \mathbb{R}^n, \;\; \forall\, \eta \in \mathbb{R}^n \;\; \nu_1 \sum_{i=1}^n \eta_i^2 \le \sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j \le \nu_2 \sum_{i=1}^n \eta_i^2, \tag{2.1}$$

$b \in L^\infty(\mathbb{R}^n)$ is non-negative, bounded, periodic function,

$$\exists\, b_1 > 0, \ \ \exists\, b_0 > 0, \ \ \forall\, s \in \mathbb{R} \ \ b_0 \le b(s) \le b_1, \tag{2.2}$$

nonlinearity $f \in C(\mathbb{R})$ satisfies the standard conditions of sign and growth:

$$\exists\, \alpha > 0, \ \ C \ge 0, \ \ p \ge 2: \ \ \forall\, s \in \mathbb{R}, \ \ f(s)\cdot s \ge \alpha|s|^p, \ \ |f(s)| \le C\big(1 + |s|^{p-1}\big), \tag{2.3}$$
$$U \text{ is convex, closed set in } L^2(Q),\ 0 \in U.$$

The function $q_\varepsilon : Q \times \mathbb{R} \longmapsto \mathbb{R}$ is a Carathéodory function, and there exist a constant $K > 0$ independent of $\varepsilon \in (0,1)$ and non-negative functions $K_1 \in L^1(Q)$, $K_2 \in L^2(Q)$ such that

$$q_\varepsilon(t,x,\xi)\xi \ge -K_1(t,x), \ \ |q_\varepsilon(t,x,\xi)| \le K|\xi| + K_2(t,x). \tag{2.4}$$

Under conditions (2.1)–(2.3), it is known [13] that for any $\varepsilon > 0$, $\forall\, u \in L^2(Q)$, $\forall\, y_0^\varepsilon \in L^2(\Omega)$, problem (1.1) has at least one solution $y = y(t,x)$ in the class

$$W := \Big\{ y \in L^2(0,\infty; H_0^1(\Omega)) : \ \frac{dy}{dt} \in L^2(0,\infty; H^{-1}(\Omega)) \Big\}.$$

Moreover, each solution (1.1) from W belongs to $C([0,\infty); L^2(\Omega))$.

Theorem 2.1. *Under conditions* (2.1)–(2.4) *the optimal control problem* (1.1)–(1.3) *has a solution* $\{\overline{y}^\varepsilon, \overline{u}^\varepsilon\}$.

Now let us discuss averaged problem ($\varepsilon = 0$).

We assume that a constant, positive defined matrix $\widehat{a}$ is averaged for $a(\frac{x}{\varepsilon})$ [10], the number $\widehat{b}$ is the mean value of a periodic function $b(x)$, and there exists a Carathéodory function $q_\varepsilon : Q \times \mathbb{R} \longmapsto \mathbb{R}$ such that

$$\forall\, r > 0 \ \ q_\varepsilon(t,x,\xi) \to q(t,x,\xi) \ \text{ weakly in } L^2(Q) \text{ uniformly with respect to } \ |\xi| \le r. \tag{2.5}$$

Consider problem (1.1)–(1.3) with averaged coefficients

$$\begin{cases} \dfrac{\partial y}{\partial t} = \operatorname{div}[\widehat{a}\nabla y] - \widehat{b} f(y) + u(t,x), \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{t=0} = y_0(x), \end{cases} \tag{2.6}$$

$$u \in U \subseteq L^2(Q_T), \tag{2.7}$$

$$J(y,u) = \int\limits_Q q(t,x,(t,x))y(t,x)\,dx + \gamma \int\limits_Q u^2(t,x)\,dt\,dx \longrightarrow \inf. \tag{2.8}$$

Using convergence (2.5), it is easy to show that the function $q_\varepsilon : Q \times \mathbb{R} \mapsto \mathbb{R}$ satisfies inequalities (2.4). Then, by Theorem 2.1, we can assert that problem (2.6)–(2.8) has a solution $\{\overline{y}, \overline{u}\}$.

We will assume the following additional condition:

$$\text{for any } \ u \in U \ \text{ problem (2.6) has a unique solution.} \tag{2.9}$$

Condition (2.9) will take place if $f \in C^1(\mathbb{R})$ and $f'(s) \ge -C_3$ [13], or $\widehat{b} \cdot f(s) \equiv 0$.

Theorem 2.2. *Let conditions* (2.1)–(2.4), (2.5), (2.9) *be satisfied, and in* (2.3) *we have*

$$p = \begin{cases} 2, & \textit{if } n \geq 3, \\ 3, & \textit{if } n = 2, \\ 4, & \textit{if } n = 1. \end{cases} \tag{2.10}$$

Let also for some number $l > 0$ the following condition be fulfilled

$$\big|q_\varepsilon(t,x,\xi_1) - q_\varepsilon(t,x,\xi_2)\big| \leq l|\xi_1 - \xi_2|. \tag{2.11}$$

Then the limit relation is true

$$J(\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}) \longrightarrow J(\overline{y}, \overline{u}), \ \varepsilon \to 0,$$

where $\{\overline{y}^{\varepsilon}, \overline{u}^{\varepsilon}\}$ and $\{\overline{y}, \overline{u}\}$ are optimal processes in problems (1.1)–(1.3) *and* (2.6)–(2.8).

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