On Some Bounds for Coefficients of the Asymptotics to Robin Eigenvalue

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Let us consider the Robin eigenvalue problem

$$\Delta u+\lambda u=0,\ \ x\in\Omega, \tag{1}$$

$$\Big(\frac{\partial u}{\partial\nu}+\alpha u\Big)\Big|_{x\in\Gamma}=0,\ \ \alpha\in\mathbb{R}. \tag{2}$$

Here $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a bounded domain with the sufficiently smooth boundary Γ. We denote by $\lambda_1^R(\alpha)$ the first eigenvalue of problem (1), (2). Consider also the Dirichlet eigenvalue problem

$$\Delta u+\lambda u=0,\ \ x\in\Omega, \tag{3}$$

$$u\big|_{x\in\Gamma}=0. \tag{4}$$

Let λ_1^D be the first eigenvalue of problem (3), (4), and $u_1^D(x)$ be the first Dirichlet eigenfunction, satisfying $\|u_1^D\|_{L_2(\Omega)}=1$.

In the papers [1–5] we get the following statement.

Theorem 1. *The eigenvalue $\lambda_1^R(\alpha)$ satisfies the asymptotic representation*

$$\lambda_1^R(\alpha)=\lambda_1^D-a_1\alpha^{-1}-a_2\alpha^{-2}+o(\alpha^{-2}),\ \ \alpha\to+\infty, \tag{5}$$

$$a_1=\int\limits_\Gamma\Big(\frac{\partial u_1^D}{\partial\nu}\Big)^2\,ds,\quad a_2=\int\limits_\Gamma\frac{\partial u_1^D}{\partial\nu}\,\frac{\partial v}{\partial\nu}\,ds. \tag{6}$$

The function $v\in H^1(\Omega)$ is a solution of the boundary value problem

$$\Delta v+\lambda_1^D v=\int\limits_\Gamma\Big(\frac{\partial u_1^D}{\partial\nu}\Big)^2\,ds\,u_1^D,\ \ x\in\Omega, \tag{7}$$

$$v\big|_{x\in\Gamma}=-\frac{\partial u_1^D}{\partial\nu}\Big|_{x\in\Gamma}, \tag{8}$$

satisfying the condition

$$\int\limits_\Omega vu_1^D\,dx=0. \tag{9}$$

Problem (7)–(9) has a unique solution.

In this paper we establish two-sided estimates for the coefficient a_1 in formula (5).

Theorem 2. *Let $\Omega \subset B_{R_0}(0) = \{x \in \mathbb{R}^n : \ |x| < R_0\}$ and $\boldsymbol{b}(x) = (b_1(x), \dots, b_n(x)) \in C^1(\overline{\Omega})$ be a vector function. Then the following estimates hold:*

$$\frac{2\lambda_1^D}{R_0} \le a_1 \le 4n \inf_{\substack{\boldsymbol{b}\in C^1(\overline{\Omega})\\ \boldsymbol{b}|_\Gamma=\nu}} \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \lambda_1^D, \ \ \|f(x)\|_{C(\overline{\Omega})} = \sup_{x\in\overline{\Omega}} |f(x)|. \tag{10}$$

Definition. We call Γ a strictly star-shaped surface if the inequality $(\nu, x) > 0$ holds for all $x \in \Gamma$.

Theorem 3. *Let Γ be a strictly star-shaped surface. Then the following estimate holds:*

$$a_1 \le \frac{2\lambda_1^D}{\inf\limits_{x\in\Gamma} (\nu, x)}\,. \tag{11}$$

Let us note that for $\Omega = B_{R_0}(0)$ it follows from (10), (11) that $a_1 = \frac{2\lambda_1^D}{R_0}$.

Proof. By direct computation we have the following equality for solutions of problem (3), (4):

$$\int\limits_\Gamma (\mathbf{b}, \nu) u_\nu^2 \, ds = \int\limits_\Omega \Big(2 \sum_{i,j=1}^n (b_i)_{x_j} u_{x_i} u_{x_j} + \operatorname{div} \mathbf{b} \big(\lambda u^2 - |\nabla u|^2 \big) \Big) \, dx. \tag{12}$$

Using (12) for $\mathbf{b}|_\Gamma = \nu$, we get

$$\int\limits_\Gamma u_\nu^2 \, ds = \int\limits_\Gamma (\mathbf{b}, \nu) u_\nu^2 \, ds \le 2 \int\limits_\Omega \sum_{i,j=1}^n (b_i)_{x_j} u_{x_i} u_{x_j} \, dx + \int\limits_\Omega |\operatorname{div} \mathbf{b}| \big(|\nabla u|^2 + \lambda u^2 \big) \, dx. \tag{13}$$

We have

$$\begin{aligned} \sum_{i,j=1}^n (b_i)_{x_j} u_{x_i} u_{x_j} &\le \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \sum_{i,j=1}^n |u_{x_i}|\, |u_{x_j}| \\ &= \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \Big(\sum_{i=1}^n |u_{x_i}| \Big)^2 \le n \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} |\nabla u|^2, \ \ x \in \Omega. \end{aligned} \tag{14}$$

Now, combine (13), (14) and the inequality

$$|\operatorname{div} \mathbf{b}| \le n \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})}, \ \ x \in \Omega,$$

we get

$$\int\limits_\Gamma u_\nu^2 \, ds \le \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \Big(3n \int\limits_\Omega |\nabla u|^2 \, dx + \lambda \int\limits_\Omega u^2 \, dx \Big). \tag{15}$$

It follows from (3), (4) that

$$\int\limits_\Omega |\nabla u|^2 \, dx = \lambda \int\limits_\Omega u^2 \, dx. \tag{16}$$

Therefore, by (15) and (16),

$$\int\limits_\Gamma u_\nu^2 \, ds \le 4n \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \lambda \int\limits_\Omega u^2 \, dx. \tag{17}$$

Taking $u = u_1^D$ with $\|u_1^D\|_{L_2(\Omega)} = 1$, we get from (6) and (17) the upper estimate (10).

Let us prove now the lower estimate (10). We have the Rellich equality for normalized in $L_2(\Omega)$ eigenfunctions of problem (3), (4) (see [6, 7]):

$$\lambda = \frac{1}{2} \int\limits_{\Gamma} (x, \nu) u_\nu^2 \, ds. \tag{18}$$

Therefore,

$$2\lambda = \int\limits_{\Gamma} (x, \nu) u_\nu^2 \, ds \le \int\limits_{\Gamma} |x| u_\nu^2 \, ds \le \sup_{x \in \Gamma} \int\limits_{\Gamma} u_\nu^2 \, ds \le R_0 \int\limits_{\Gamma} u_\nu^2 \, ds.$$

Now, for $u = u_1^D$ we obtain

$$a_1 \ge \frac{2\lambda_1^D}{R_0}. \qquad \square$$

The proof of Theorem 3 is based on the Rellich equality (18) for u_1^D in a strictly star-shaped domain Ω.

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