On an Upper Estimate for the First Eigenvalue of a Sturm–Liouville Problem

S. Ezhak, M. Telnova

Plekhanov Russian University of Economics, Moscow, Russia
E-mail: svetlana.ezhak@gmail.com; mytelnova@yandex.ru

1 Introduction

Consider the Sturm–Liouville problem

$$y''+Q(x)y+\lambda y=0,\ \ x\in(0,1), \tag{1.1}$$

$$y(0)=y(1)=0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all measurable locally integrable on $(0,1)$ functions with non-negative values such that the following integral conditions hold

$$\int\limits_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx=1,\ \ \alpha,\beta,\gamma\in\mathbb{R},\ \ \gamma\neq 0, \tag{1.3}$$

$$\int\limits_0^1 x(1-x)Q(x)\,dx<\infty.$$

A function y is a *solution to problem* (1.1), (1.2) if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho,1-\rho]$, where $0<\rho<\frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

This work is a continuation of studies of estimates for the first eigenvalue of the Sturm–Liouville problem with the equation $y''+\lambda Q(x)y=0$, Dirichlet boundary conditions, and a non-negative summable on $[0,1]$ potential Q satisfying the condition $\|Q\|_{L_{\gamma}(0,1)}=1$, $\gamma\neq 0$, initiated by Y. V. Egorov and V. A. Kondratiev in [1]. We study a problem of that kind provided the integral conditions contain weight functions.

For $\gamma<0$, $\alpha\leq 2\gamma-1$, $-\infty<\beta<+\infty$ or $\gamma<0$, $\beta\leq 2\gamma-1$, $-\infty<\alpha<+\infty$, the set $T_{\alpha,\beta,\gamma}$ is empty and the first eigenvalue of problem (1.1), (1.2) does not exist. For other values of $\alpha,\beta,\gamma,\gamma\neq 0$, denote

$$M_{\alpha,\beta,\gamma}=\sup_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q).$$

2 Main results

It [2], the following theorem was proved.

Theorem 2.1. *If $0<\gamma<1$, $\alpha,\beta>2\gamma-1$, then $M_{\alpha,\beta,\gamma}<\pi^2$.*

In the proof of this theorem it was supposed that for any $0<\gamma<1$, $\alpha,\beta>2\gamma-1$ we have $M_{\alpha,\beta,\gamma}=\pi^2$, that is, for any sufficiently small $\varepsilon>0$ there exists a function $Q\in T_{\alpha,\beta,\gamma}$ such that $\lambda_1(Q)>(\pi-\varepsilon)^2$. Under this assumption we got a contradiction with condition (1.3), namely, it was proved that in this case there exists a positive constant C, depending on α, β, γ, such that

$$\int\limits_0^1 x^\alpha(1-x)^\beta Q^\gamma(x)dx\le C\varepsilon^M,$$

where

$$M=\min\Big\{\frac{(\alpha-2\gamma+1)\gamma}{1-\gamma+\alpha}\,,\,\frac{(\beta-2\gamma+1)\gamma}{1-\gamma+\beta}\Big\}>0.$$

Let us prove the following

Theorem 2.2. *If $\alpha,\beta>1$, then $M_{\alpha,\beta,1}<\pi^2$.*

Proof. Suppose that $M_{\alpha,\beta,1}=\pi^2$, $\alpha,\beta>1$.

Let $0<\gamma<1$. By the Hölder inequality, we have

$$\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^\gamma(x)\,dx\leqslant\bigg(\int\limits_0^1 x^\alpha(1-x)^\beta Q(x)\,dx\bigg)^\gamma.$$

Then

$$\bigg(\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^\gamma(x)\,dx\bigg)^{\frac1\gamma}\leqslant\int\limits_0^1 x^\alpha(1-x)^\beta Q(x)\,dx=1. \tag{2.1}$$

Note that, for $0<\gamma<1$, the inequality $\alpha\gamma>2\gamma-1$ holds if and only if $\alpha>1$. Similarly, $\beta\gamma>2\gamma-1$ if and only if $\beta>1$.

Denote by $\widetilde{T}_{\alpha\gamma,\beta\gamma,\gamma}$ a set of measurable non-negative locally integrable on $(0,1)$ functions Q such that

$$\bigg(\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^\gamma(x)\,dx\bigg)^{\frac1\gamma}\leqslant 1.$$

By virtue of (2.1),

$$T_{\alpha,\beta,1}\subset\widetilde{T}_{\alpha\gamma,\beta\gamma,\gamma}.$$

If we suppose that

$$M_{\alpha,\beta,1}=\sup_{Q\in T_{\alpha,\beta,1}}\lambda_1(Q)=\pi^2,$$

then for $0<\gamma<1$, we also have

$$M_{\alpha\gamma,\beta\gamma,\gamma}=\sup_{Q\in\widetilde{T}_{\alpha\gamma,\beta\gamma,\gamma}}\lambda_1(Q)=\pi^2.$$

If $M_{\alpha,\beta,1}=\sup\limits_{Q\in T_{\alpha,\beta,1}}\lambda_1(Q)=\pi^2$, then for any $\varepsilon>0$ there exists a function $Q_*\in T_{\alpha,\beta,1}$ such that

$$\lambda_1(Q_*)>(\pi-\varepsilon)^2.$$

This function Q_* belongs to $\widetilde{T}_{\alpha\gamma,\beta\gamma,\gamma}$ either and following the proof of Theorem 2.1, we can find a positive constant C, depending on α,β,γ, such that

$$\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}{Q_*}^{\gamma}(x)\,dx \leqslant C\varepsilon^M,$$

where

$$M=\min\Big\{\frac{(\alpha\gamma-2\gamma+1)\gamma}{1-\gamma+\alpha\gamma}\,,\frac{(\beta\gamma-2\gamma+1)\gamma}{1-\gamma+\beta\gamma}\Big\}.$$

Suppose that $M=\frac{(\alpha\gamma-2\gamma+1)\gamma}{1-\gamma+\alpha\gamma}$ (in case $M=\frac{(\beta\gamma-2\gamma+1)\gamma}{1-\gamma+\beta\gamma}$ the proof is similar). For any $0<\gamma<1$, $\alpha>1$, we have $\alpha\gamma>2\gamma-1$ and M is positive.

Note that for a fixed $\alpha>1$, if γ approaches 1, the exponent of $\varepsilon^{\frac{(\alpha\gamma-2\gamma+1)\gamma}{1-\gamma+\alpha\gamma}}$ approaches a concrete positive number $\frac{\alpha-1}{\alpha}$.

As soon as γ tends to 1, the factor C, depending on α,β,γ, tends to some constant $\widetilde{C}$. Let us choose ε in such a way that the following inequality

$$\widetilde{C}\varepsilon^{\frac{\alpha-1}{\alpha}}<\frac{1}{2}$$

holds.

Then we get a contradiction

$$1=\int\limits_0^1 x^{\alpha}(1-x)^{\beta}Q(x)\,dx=\lim_{\gamma\to1}\bigg(\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^{\gamma}(x)\,dx\bigg)^{\frac{1}{\gamma}}\leqslant\widetilde{C}\varepsilon^{\frac{\alpha-1}{\alpha}}<\frac{1}{2}\,.$$

Note that, while γ increases from 0 to 1, the integral $\big(\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^{\gamma}(x)\,dx\big)^{\frac{1}{\gamma}}$ also increases.

Indeed, if $\gamma_1<\gamma_2$, then by virtue of the Hölder inequality, since $\frac{\gamma_2}{\gamma_1}>1$, we have

$$\int\limits_0^1 x^{\alpha\gamma_1}(1-x)^{\beta\gamma_1}Q^{\gamma_1}(x)\,dx\leqslant\bigg(\int\limits_0^1 x^{\alpha\gamma_2}(1-x)^{\beta\gamma_2}Q^{\gamma_2}(x)\,dx\bigg)^{\frac{\gamma_1}{\gamma_2}}$$

and

$$\bigg(\int\limits_0^1 x^{\alpha\gamma_1}(1-x)^{\beta\gamma_1}Q^{\gamma_1}(x)\,dx\bigg)^{\frac{1}{\gamma_1}}\leqslant\bigg(\int\limits_0^1 x^{\alpha\gamma_2}(1-x)^{\beta\gamma_2}Q^{\gamma_2}(x)\,dx\bigg)^{\frac{1}{\gamma_2}}.\qquad\Box$$

References

[1] Yu. Egorov and V. Kondratiev, *On Spectral Theory of Elliptic Operators*. Operator Theory: Advances and Applications, 89. Birkhäuser Verlag, Basel, 1996.

[2] S. Ezhak and M. Telnova, On conditions on the potential in a Sturm–Liouville problem and an upper estimate of its first eigenvalue. *Springer Proceedings in Mathematics & Statistics, International Conference on Differential & Difference Equations and Applications ICDDEA 2019*, pp. 481–496, *Differential and Difference Equations with Applications*, 2019.