On Asymptotics of Rapidly Varying Solutions of Non-Autonomous Differential Equations of Third-Order

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Consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1,1\}$, $p : [a,\omega[\to]0,+\infty[$ is a continuous function, $y < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0,+\infty[$ is a continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ for } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to 0\\ y\in\Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{or} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 equals either zero or $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of Y_0.

From the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} + 1 \ \text{ for } \ y \in \Delta_{Y_0}$$

and conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ for } \ y \to Y_0 \quad \big(y \in \Delta_{Y_0}\big) \ \text{ and } \ \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

This means that in the considered equation the continuous function φ and its first order derivatives are (see [8, Ch. 3, § 3.4, Lemmas 3.2, 3.3, pp. 91–92]) rapidly changing as $y \to Y_0$.

For two-term differential equations of the form (1) with nonlinearities satisfying condition (2), the asymptotic properties of solutions were studied in the works of M. Maric [8], V. M. Evtukhov and his students N. G. Drik, V. M. Kharkov, A. G. Chernikova [3–5].

In the works of V. M. Evtukhov, A. G. Chernikova [3] for the differential equation (1) of the second order in the case, when φ is a rapidly changing function as $t \to +\infty$, the asymptotic properties of the so-called $P_\omega(Y_0,\lambda_0)$-solutions were studied. In this work, we propose the distribution of these results to third-order differential equations.

Definition 1. Solution y of equation (1) is called $P_\omega(Y_0,\lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is specified on the interval $[t_0,\omega[\subset [a,\omega[$ and satisfies the following conditions

$$y(t) \in \Delta_{Y_0}, \ \text{ where } \ t \in [t_0,\omega[\,,$$

$$\lim_{t\uparrow\omega} y(t) = Y_0, \quad \lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad k = 1,2, \quad \lim_{t\uparrow\omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The goal of this work is to establish the necessary and sufficient conditions for the existence for equation (1) of (Y_0,λ_0)-solutions in the non-singular case, when $\lambda_0\in\mathbb{R}\setminus\{0,1,\frac{1}{2}\}$, and in the singular case, when $\lambda_0=1$, as well as asymptotic for $t\uparrow\omega$ representations for such solutions and their derivatives up to the second order.

Without loss of generality, we will further assume that

$$\Delta_{Y_0}=\begin{cases}[y_0,Y_0[& \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0,y_0] & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0,\end{cases}$$

where $y_0\in\mathbb{R}$ is such that $|y_0|<1$ when $Y_0=0$, and $y_0>1$ ($y_0<-1$), when $Y_0=+\infty$ (when $Y_0=-\infty$).

The function $f:\Delta_{Y_0}\to\mathbb{R}\setminus\{0\}$, satisfying condition (2), when $Y_0=\pm\infty$, and $\lim\limits_{y\to+\infty}f(y)=+\infty$, belongs to the class $\Gamma_{Y_0}(Z_0)$ of the functions $\varphi:\Delta_{Y_0}\to]0,+\infty[$, where Y_0 equals either zero or $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of Y_0, for which

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\varphi(y)=Z_0=\begin{cases} & 0,\\ \text{or} & +\infty,\end{cases}\tag{3}$$

which extends the class of function Γ, introduced by L. Khan (see, for example, [6, Ch. 3, p. 3.10, p. 175]).

If $f\in\Gamma_{Y_0}(Z_0)$ with the complementary function g, and, moreover, is continuous and strictly monotone, then there exists a continuous strictly monotone inverse function $f^{-1}:\Delta_{Z_0}\longrightarrow\Delta_{Y_0}$, where

$$\Delta_{Z_0}=\begin{cases} & [z_0,Z_0[\,,\\ \text{or} &]Z_0,z_0],\end{cases}\quad z_0=f(y_0),\quad Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}f(y).$$

We introduce the necessary auxiliary notation. We assume that the domain of the function φ in equation (1) is determined by formula (3). Next, we set

$$\mu_0=\operatorname{sign}\varphi'(y),\quad \nu_0=\operatorname{sign}y_0,\quad \nu_1=\begin{cases}1 & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1 & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the functions

$$J(t)=\int\limits_A^t\pi_\omega^2(\tau)p(\tau)\,d\tau,\quad \Phi(y)=\int\limits_B^y\frac{ds}{\varphi(s)}\,,$$

where

$$\pi_\omega=\begin{cases}t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}$$

$$A=\begin{cases}\omega & \text{if } \displaystyle\int\limits_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=const,\\ a & \text{if } \displaystyle\int\limits_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=\infty,\end{cases}\qquad B=\begin{cases}Y_0 & \text{if } \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=const,\\ y_0 & \text{if } \int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=const.\end{cases}$$

Considering the definition of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1), we note that the numbers ν_0, ν_1, ν_2 and α_0 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solutions of its first, second and third derivatives (respectively) in some left neighborhood of ω. It is clear that the condition

$$\nu_0\nu_1<0, \ \text{ if } \ Y_0=0, \quad \nu_0\nu_1>0, \ \text{ if } \ Y_0=\pm\infty,$$

is necessary for the existence of such solutions.

Now we turn our attention to some properties of the function Φ. It retains a sign on the interval Δ_{Y_0}, tends either to zero or $\pm\infty$ when $y\to Y_0$ and increasing by Δ_{Y_0}, because on this interval $\Phi'(y)=\frac{1}{\varphi(y)}>0$. Therefore, for it there is an inverse function $\Phi^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where due to the second of conditions (2) and the monotone increase of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases} \quad 0, \\ \text{or} \quad +\infty,\end{cases} \qquad \Delta_{Z_0}=\begin{cases}[z_0,Z_0[& \text{for } \ \Delta_{Y_0}=[y_0,Y_0[\,,\\ ]Z_0,z_0] & \text{for } \ \Delta_{Y_0}=]Y_0,y_0],\end{cases} \qquad z_0=\varphi(y_0).$$

For $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac12\}$ we also introduce auxiliary functions:

$$q(t)=\frac{\alpha_0(\lambda_0-1)^2\pi_\omega^3(t)p(t)\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,(\lambda_0-1)J(t))\big)}{\lambda_0\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}\,,$$

$$H(t)=\frac{\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,,$$

For equation (1) the following assertions take place.

Theorem 1. *Let $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac12\}$. Then for the existence for the differential equation* (1) *of $P_\omega(Y_0,\lambda_0)$-solutions, it is necessary to comply with the conditions*

$$\alpha_0\nu_1\lambda_0>0, \quad \nu_0\nu_1(2\lambda_0-1)(\lambda_0)\pi_\omega(t)>0 \ \text{ and } \ \alpha_0\mu_0\lambda_0J(t)<0 \ \text{ for } \ t\in(a,\omega), \tag{4}$$

$$\frac{\alpha_0}{\lambda_0}\lim_{t\uparrow\omega}J(t)=Z_0, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'(t)}{J(t)}=\pm\infty, \quad \lim_{t\uparrow\omega}q(t)=\frac{2\lambda_0-1}{\lambda_0-1}\,. \tag{5}$$

Moreover, for each such solution, the following asymptotic representations take place:

$$y(t)=\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big] \ \text{ for } \ t\uparrow\omega, \tag{6}$$

$$y'(t)=\frac{(2\lambda_0-1)}{(\lambda_0-1)}\,\frac{\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}{\pi_\omega(t)}\,[1+o(1)] \ \text{ for } \ t\uparrow\omega, \tag{7}$$

$$y''(t)=\frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2}\,\frac{\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}{\pi_\omega^2(t)}\,[1+o(1)] \ \text{ for } \ t\uparrow\omega.$$

Theorem 2. *Let $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac12\}$, conditions* (4), (5) *met, there exist a finite or equal to $\pm\infty$ limit*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2}\,\sqrt[3]{\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^2}\,,$$

and there exist the limit

$$\lim_{t\uparrow\omega}\Big[\frac{2\lambda_0-1}{\lambda_0-1}-q(t)\Big]|H(t)|^{\frac23}=0,$$

Then, the differential equation (1) *has at least one* $P_\omega(Y_0,\lambda_0)$*-solution, which allows for* $t\uparrow\omega$ *the asymptotic representations*

$$\begin{aligned}
y(t) &= \Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big],\\
y'(t) &= \frac{2\lambda_0-1}{(\lambda_0-1)\pi_\omega(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+o(1)H^{-\frac{2}{3}}],\\
y''(t) &= \frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)\pi_\omega^2(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+o(1)H^{-\frac{1}{3}}],
\end{aligned}\tag{8}$$

and in the case when

$$\mu_0\lambda_0(2\lambda_0-1)(\lambda_0-1)<0 \ \ \textit{for} \ \ t\in(a,\omega),$$

the differential equation (1) *has a one-parameter family of* $P_\omega(Y_0,\lambda_0)$*-solutions, but in the case when*

$$\mu_0\lambda_0(2\lambda_0-1)(\lambda_0-1)>0 \ \ \textit{for} \ \ t\in(a,\omega),$$

the differential equation (1) *has a two-parameter family of* $P_\omega(Y_0,\lambda_0)$*-solutions with representations* (6), (7), *and such that the first and second order derivatives allow the asymptotic representations* (8).

Introduce the functions

$$J_1(t)=\int\limits_{A_1}^{t} p^{\frac{1}{3}}(\tau)\,d\tau, \quad \Phi_1(y)=\int\limits_{B_1}^{y}\frac{ds}{|s|^{\frac{2}{3}}\varphi^{\frac{1}{3}}(s)}\,,$$

where

$$A_1=\begin{cases} \omega & \text{if} \ \displaystyle\int\limits_a^\omega p^{\frac{1}{3}}(\tau)\,d\tau<+\infty,\\ a & \text{if} \ \displaystyle\int\limits_a^\omega p^{\frac{1}{3}}(\tau)\,d\tau=+\infty,\end{cases} \qquad B_1=\begin{cases} Y_0 & \text{if} \ \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac{2}{3}}\varphi^{\frac{1}{3}}(s)}=const,\\ y_0 & \text{if} \ \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac{2}{3}}\varphi^{\frac{1}{3}}(s)}=\pm\infty.\end{cases}$$

Consider the definition of $P_\omega(Y_0,1)$-solutions of the differential equation (1). It is clear that the conditions

$$\nu_0\nu_1<0, \ \text{if} \ Y_0=0, \quad \nu_0\nu_1>0, \ \text{if} \ Y_0=\pm\infty,$$

and

$$\nu_1\alpha_0<0, \ \text{for} \ \lim_{t\uparrow\omega}y'(t)=0, \quad \nu_1\alpha_0>0, \ \text{for} \ \lim_{t\uparrow\omega}y'(t)=\pm\infty,$$

are necessary for the existence of such solutions. For $\lambda_0=1$, we also introduce the auxiliary functions

$$\begin{aligned}
q_1(t) &= \frac{\alpha_0\nu_1 J_3(t)}{p^{\frac{1}{3}}(t)\Phi_1^{-1}(\nu_1J_1(t))^{\frac{2}{3}}\varphi^{\frac{1}{3}}(\Phi_1^{-1}(\nu_1J_1(t)))}\,,\\
H_1(t) &= \frac{\Phi_1^{-1}(\nu_1J_1(t))\varphi'(\Phi_1^{-1}(\nu_1J_1(t)))}{\varphi(\Phi_1^{-1}(\nu_1J_1(t)))}\,,\\
J_2(t) &= \int\limits_{A_2}^{t} p(\tau)\varphi(\Phi_1^{-1}(\nu_1J_1(\tau)))\,d\tau, \quad J_3(t)=\int\limits_{A_3}^{t} J_2(\tau)\,d\tau,
\end{aligned}$$

where

$$A_2 = \begin{cases} t_0 & \text{if } \int\limits_{t_2}^{\omega} p(\tau)\varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau = +\infty, \\ \omega & \text{if } \int\limits_{t_2}^{\omega} p(\tau)\varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau < +\infty, \end{cases} \qquad A_3 = \begin{cases} t_0 & \text{if } \int\limits_{t_3}^{\omega} J_2(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \int\limits_{t_3}^{\omega} J_2(\tau)\,d\tau < +\infty, \end{cases}$$

$$t_2, t_3 \in [a, \omega].$$

For equation (1) the following assertions take place.

Theorem 3. *For the existence for the differential equation* (1) *of* $P_\omega(Y_0, 1)$*-solutions it is necessary to comply with the conditions*

$$\alpha_0\nu_0 > 0, \ \ \mu_0\nu_1 J_1(t) < 0 \ \ \text{for} \ \ t \in]a, \omega[\,,$$

$$\nu_1 \lim_{t\uparrow\omega} J_1(t) = Z_0, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) J_1'(t)}{J_1(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} q_1(t) = 1, \ \ \lim_{t\uparrow\omega} \frac{p(t)\varphi(\Phi_1^{-1}(\nu_1 J_1(t))) J_3(t)}{(J_2(t))^2} = 1.$$

Moreover, for each solution, there take place the asymptotic representations for $t \uparrow \omega$

$$y(t) = \Phi_1^{-1}\big(\alpha_0(\lambda_0 - 1) J_1(t)\big)\Big[1 + \frac{o(1)}{H_1(t)}\Big],$$
$$y'(t) = \nu_1 p^{\frac{1}{3}}(t)\varphi^{\frac{1}{3}}\big(\Phi_1^{-1}(\nu_1 J_1(t))\big)\big(\Phi_1^{-1}(\nu_1 J_1(t))\big)^{\frac{2}{3}}[1 + o(1)],$$
$$y''(t) = \alpha_0 J_2(t)[1 + o(1)].$$

Similarly to Theorem 2, we prove a sufficient condition for the existence of $P_\omega(Y_0, 1)$-solutions.

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