Asymptotic Properties of Solutions to Half-Linear Differential Equation with Negative Potential

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1 Introduction

We present a complete overview on the qualitative behavior of solutions of the half-linear equation

$$(a(t)|x'|^{\alpha}\operatorname{sgn} x')' - b(t)|x|^{\alpha}\operatorname{sgn} x = 0, \tag{1.1}$$

where α is a positive constant and the functions a, b are continuous and positive for $t \geq t_0 \geq 0$.

Equation (1.1) comes out in studying radial solutions of equations with p-Laplacian operator and have been widely investigated in the literature.

The study on the qualitative behavior of solutions of (1.1), especially as concerns the classification of solutions, the existence of monotone bounded or unbounded solutions, the growth at infinity or the decay at zero of solutions, has a long history. Many of the results obtained in these fields have been obtained for more general equations and it would be impossible to mention all of them. As regards in particular the half-linear case, we recall the pioneering works of Elbert and Mirzov [13, 25] and we refer the reader for more details to the monographs [12, 26] and references therein.

Interesting contributions are due to the Georgian and Russian mathematical school. Almost all of these papers concern very general differential equations, which include, in particular, the Emden–Fowler equation or the Thomas-Fermi equation, see [4, 6, 7, 16, 18, 22, 28]. Recently, other developments are given by the Japanese mathematical school, see [14,15,19,20,27,30] under different point of view.

Our aim here is to present a complete overview, *quadro completo*, to the asymptotic behavior of solutions of (1.1). This result is a generalization of the one corresponding for the linear equation with Sturm–Liouville differential operator, see, e.g. [5]. In particular, we show that when the functions a, b have, roughly speaking, a power behavior near infinity, then the complete overview is the same as for the linear equation. Our approach follows that one in [1–3], even if here the results are more complete and the method is slightly different.

2 Basic properties

Following the linear case, we introduce a classification of solutions of (1.1), which is based on the one in [1–3], with minor modifications. We note that a slightly different classification of solutions of an equation, which includes (1.1), has been used in [27] under the additional assumption $a^{-1/\alpha} \notin L^1[t_0,\infty)$ and in [30] in the opposite situation $a^{-1/\alpha} \in L^1[t_0,\infty)$. Passing from the linear case to the half-linear one, it is well-known that several basic differences arise, see, e.g., [12, Section 1.3]. In particular, the set of solutions of (1.1), $\alpha \neq 1$, is not a linear space, the Jacobi–Liouville identity for the Wronskian and the variation of constant principle fail to hold for (1.1) with $\alpha \neq 1$. Independently of this fact, there are also a lot of similarities in asymptotic behavior of solutions.

Recall that any nontrivial solution x of (1.1) is defined on the whole interval $[t_0,\infty)$ and satisfies $\sup\limits_{t\in[\tau,\infty)} |x(t)| > 0$ for any $\tau \geq t_0$. Moreover, the Cauchy problem for (1.1) is uniquely solvable for any couple of initial data. In other words, given $T \geq t_0$ and $x_0, x_1 \in \mathbb{R}$, there exists a unique solution x of (1.1) satisfying $x(T) = x_0$, $x'(T) = x_1$ and x is defined on the whole interval $[t_0,\infty)$. Consequently, $x \equiv 0$ if and only if $x_0 = x_1 = 0$. Further, equation (1.1) is disconjugate on $[t_0,\infty)$, that is any nontrivial solution of (1.1) has at most one zero on $[t_0,\infty)$. Hence, (1.1) is nonoscillatory. The following holds.

Theorem 2.1. *The set of nontrivial solutions of* (1.1) *may be divided into two classes*

$$\mathbb{M}^+ = \big\{x \textit{ solution of } (1.1):\ \exists\, t_x \geq t_0:\ x(t)x'(t) > 0 \textit{ for } t > t_x\big\},$$
$$\mathbb{M}^- = \big\{x \textit{ solution of } (1.1):\ x(t)x'(t) < 0 \textit{ for } t > t_0\big\},$$

and both classes are nonempty. In particular, solutions x of (1.1)*, satisfying either $x(T) = 0$, $x'(T) > 0$ or $x(T) > 0$, $x'(T) = 0$ at some $T \geq t_0$, are positive increasing on (T,∞) and belong to the class $\mathbb{M}^+$. Further, if a solution of* (1.1) *in the class $\mathbb{M}^+$ is bounded, then every solution in the class $\mathbb{M}^+$ is bounded, too.*

The proof of Theorem 2.1 follows an idea used by Mambriani to solve the well-known Thomas-Fermi problem, see [29, Chapter XII, Section 5.]. An alternative proof can be found in [7].

The asymptotic behavior of solutions of (1.1) depends on the four integrals

$$J_1 = \int\limits_{t_0}^{\infty} a^{-1/\alpha}(t)\bigg(\int\limits_{t_0}^{t} b(r)\,dr\bigg)^{1/\alpha} dt, \quad J_2 = \int\limits_{t_0}^{\infty} a^{-1/\alpha}(t)\bigg(\int\limits_{t}^{\infty} b(r)\,dr\bigg)^{1/\alpha} dt,$$
$$Y_1 = \int\limits_{t_0}^{\infty} b(t)\bigg(\int\limits_{t}^{\infty} a^{-1/\alpha}(r)\,dr\bigg)^{\alpha} dt, \quad Y_2 = \int\limits_{t_0}^{\infty} b(t)\bigg(\int\limits_{t_0}^{t} a^{-1/\alpha}(r)\,dr\bigg)^{\alpha} dt.$$

A complete classification of solutions (1.1) require a preliminary analysis of mutual behavior of these integrals. Using some integral inequalities, we get the following.

Lemma 2.1 ([11]). *If $\alpha \geq 1$, then*

$$Y_2 = \infty \implies J_2 = \infty, \quad Y_1 = \infty \implies J_1 = \infty.$$

If $0 < \alpha \leq 1$, then

$$J_2 = \infty \implies Y_2 = \infty, \quad J_1 = \infty \implies Y_1 = \infty.$$

Lemma 2.1 can be viewed as an extension of the Fubini theorem. Indeed, when $\alpha = 1$, we have

$$J_1 = Y_1, \quad J_2 = Y_2. \tag{2.1}$$

By virtue of Lemma 2.1, the possible cases concerning the convergence of integrals $J_i, Y_i, i = 1, 2$, are the following eight:

(C_1): $J_1 = \infty$, $J_2 = \infty$, $Y_1 = \infty$, $Y_2 = \infty$, $\alpha > 0$;

(C_2): $J_1 = \infty$, $J_2 < \infty$, $Y_1 = \infty$, $Y_2 < \infty$, $\alpha > 0$;

(C_3): $J_1 < \infty$, $J_2 = \infty$, $Y_1 < \infty$, $Y_2 = \infty$, $\alpha > 0$;

(C_4): $J_1 < \infty$, $J_2 < \infty$, $Y_1 < \infty$, $Y_2 < \infty$, $\alpha > 0$.

(C_5): $J_1 = \infty$, $J_2 = \infty$, $Y_1 = \infty$, $Y_2 < \infty$, $\alpha > 1$;

(C_6): $J_1 = \infty$, $J_2 = \infty$, $Y_1 < \infty$, $Y_2 = \infty$, $\alpha > 1$;

(C_7): $J_1 = \infty$, $J_2 < \infty$, $Y_1 = \infty$, $Y_2 = \infty$, $0 < \alpha < 1$;

(C_8): $J_1 < \infty$, $J_2 = \infty$, $Y_1 = \infty$, $Y_2 = \infty$, $0 < \alpha < 1$.

All cases (C_n), $n = 1, \dots, 8$, may occur, as examples below. Cases (C_1)–(C_4), may occur for any $\alpha > 0$. Cases (C_5) and (C_6) may occur only when $\alpha > 1$, and cases (C_7), (C_8) only when $0 < \alpha < 1$. Thus, cases (C_5)–(C_8), do not occur in the linear case and so, roughly speaking, they are typical for the half-linear case. When $\alpha = 1$, that is for the linear equation, the possible cases are only the four cases (C_1)–(C_4). Moreover, in view of (2.1), for the linear equation the integrals Y_1 and Y_2 do not play any role.

3 A complete overview

A precise and complete classification of solutions x of (1.1) may be done by considering also the asymptotic behavior of the quasiderivative $x^{[1]}$, that is the function $x^{[1]}(t) = a(t)|x'(t)|^\alpha \operatorname{sgn} x'(t)$.

Any solution of (1.1) in the class $\mathbb{M}^+$ belongs to one of the following four subclasses:

$$\begin{aligned}
\mathbb{M}^+_{\infty,\infty} &= \Big\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |x^{[1]}(t)| = \infty\Big\},\\
\mathbb{M}^+_{\infty,\ell} &= \Big\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |x^{[1]}(t)| < \infty\Big\},\\
\mathbb{M}^+_{\ell,\infty} &= \Big\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| < \infty, \ \lim_{t\to\infty} |x^{[1]}(t)| = \infty\Big\},\\
\mathbb{M}^+_{\ell,\ell} &= \Big\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| < \infty, \ \lim_{t\to\infty} |x^{[1]}(t)| < \infty\Big\}.
\end{aligned}$$

Similarly, any solution of (1.1) in the class $\mathbb{M}^-$ belongs to one of the following four subclasses:

$$\begin{aligned}
\mathbb{M}^-_{\ell,\ell} &= \Big\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) \neq 0, \ \lim_{t\to\infty} x^{[1]}(t) \neq 0\Big\},\\
\mathbb{M}^-_{\ell,0} &= \Big\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) \neq 0, \ \lim_{t\to\infty} x^{[1]}(t) = 0\Big\},\\
\mathbb{M}^-_{0,\ell} &= \Big\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} x^{[1]}(t) \neq 0\Big\},\\
\mathbb{M}^-_{0,0} &= \Big\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} x^{[1]}(t) = 0\Big\}.
\end{aligned}$$

Unbounded solutions x of (1.1) are also called either *strongly increasing* (as $t \to \infty$) or *regular increasing* (as $t \to \infty$), according to $x \in \mathbb{M}^+_{\infty,\infty}$ or $x \in \mathbb{M}^+_{\infty,\ell}$, respectively. Such a terminology originates from the Georgian mathematical school, see [18, 22]. Indeed, when $a(t) \equiv 1$, for any unbounded eventually positive solutions x, we have either

$$\lim_{t\to\infty} \frac{x(t)}{t} = \infty \ \text{ or } \ \lim_{t\to\infty} \frac{x(t)}{t} = \ell_x, \ \ 0 < \ell_x < \infty,$$

according to x is strongly increasing or regular increasing, respectively. Analogously, solutions x of (1.1) such that $\lim_{t\to\infty} x(t)=0$ are called either *strongly decaying* or *regular decaying* (as $t\to\infty$), according to $x\in\mathbb{M}^-_{0,0}$, or $x\in\mathbb{M}^-_{0,\ell}$, respectively. Sometimes, solutions in the subclasses $\mathbb{M}^+_{\infty,\infty}$ and $\mathbb{M}^-_{0,0}$ are called *extremal solutions*.

The following result gives a complete overview of the asymptotic behavior of solutions of (1.1).

Theorem 3.1. *For the half-linear equation* (1.1) *we have* $\mathbb{M}^+\neq\varnothing$, $\mathbb{M}^-\neq\varnothing$. *Further,*

(1) *If* (C_1) *holds, then* $\mathbb{M}^+=\mathbb{M}^+_{\infty,\infty}\neq\varnothing$ *and* $\mathbb{M}^-=\mathbb{M}^-_{0,0}\neq\varnothing$.

(2) *If* (C_2) *holds, then* $\mathbb{M}^+=\mathbb{M}^+_{\infty,\ell}\neq\varnothing$ *and* $\mathbb{M}^-=\mathbb{M}^-_{\ell,0}\neq\varnothing$.

(3) *If* (C_3) *holds, then* $\mathbb{M}^+=\mathbb{M}^+_{\ell,\infty}\neq\varnothing$ *and* $\mathbb{M}^-=\mathbb{M}^-_{0,\ell}\neq\varnothing$.

(4) *If* (C_4) *holds, then* $\mathbb{M}^+=\mathbb{M}^+_{\ell,\ell}\neq\varnothing$, $\mathbb{M}^-_{\ell,\ell}\neq\varnothing$, $\mathbb{M}^-_{\ell,0}\neq\varnothing$, $\mathbb{M}^-_{0,\ell}\neq\varnothing$ *and* $\mathbb{M}^-_{0,0}=\varnothing$.

(5) *If* (C_5) *holds, then* $\mathbb{M}^+=\mathbb{M}^+_{\infty,\ell}\neq\varnothing$ *and* $\mathbb{M}^-=\mathbb{M}^-_{0,0}\neq\varnothing$.

(6) *If* (C_6) *holds, then* $\mathbb{M}^+=\mathbb{M}^+_{\infty,\infty}\neq\varnothing$ *and* $\mathbb{M}^-=\mathbb{M}^-_{0,\ell}\neq\varnothing$.

(7) *If* (C_7) *holds, then* $\mathbb{M}^+=\mathbb{M}^+_{\infty,\infty}\neq\varnothing$ *and* $\mathbb{M}^-=\mathbb{M}^-_{\ell,0}\neq\varnothing$.

(8) *If* (C_8) *holds, then* $\mathbb{M}^+=\mathbb{M}^+_{\ell,\infty}\neq\varnothing$ *and* $\mathbb{M}^-=\mathbb{M}^-_{0,0}\neq\varnothing$.

A complete proof of Theorem 3.1 can be found in a forthcoming monograph [8, Chapter V]. It is based on several tools. In particular, we use the Tychonoff fixed point theorem, certain functional integral inequalities jointly with a comparison between (1.1) and the equation

$$\Big(\frac{1}{b^{1/\alpha}(t)}\,|z'|^{1/\alpha}\operatorname{sgn} z'\Big)'-\frac{1}{a^{1/\alpha}(t)}\,|z|^{1/\alpha}\operatorname{sgn} z=0, \tag{3.1}$$

which comes from (1.1) replacing a by $b^{-1/\alpha}$, b by $a^{-1/\alpha}$ and α with α^{-1}. Equation (3.1) is called *reciprocal equation* to (1.1) and its role in studying the qualitative behavior of solutions of (1.1) is described by the *Reciprocity Principle*, see, e.g., [2, 12]. In particular, observe that the integrals J_1 and J_2 read for (3.1) as Y_2 and Y_1, respectively. Further, also some interesting properties of solutions of (1.1) are also used in the proof, like the property that two solutions of (1.1) can cross at most at one point $T\geq t_0$, whereby the case $T=\infty$ is included, when these solutions are bounded.

Theorem 3.1 extends [3, Theorem 1] by giving the complete classification of solutions. Alternative proofs of some claims of Theorem 3.1 can be found in [3, Theorem 1], too.

4 Examples

Example 4.1. Consider the half-linear equation (1.1) and let there exist $\mu,\nu\in\mathbb{R}$ such that

$$\lim_{t\to\infty}\frac{a(t)}{t^{\mu}}=a_\infty,\quad \lim_{t\to\infty}\frac{b(t)}{t^{\nu}}=b_\infty,\ \ a_\infty,b_\infty\in(0,\infty). \tag{4.1}$$

Using a standard calculation and Lemma 2.1 we have

$$J_1=\infty\iff Y_1=\infty\ \text{ and }\ J_2=\infty\iff Y_2=\infty.$$

Consequently, when (4.1) holds, the possible cases concerning the convergence of integrals J_i, Y_i, $i=1,2$, are the four cases (C_1)–(C_4). In other words, in this case the integrals Y_1, Y_2 do not play any role. Hence, the situation is exactly the one which happens in the linear case.

Example 4.2. Consider the half-linear equation for $t \geq t_0 > 0$,

$$\big(e^{-3t}(x')^3\big)' - t^{-2}e^{-3t}x^3 = 0. \tag{4.2}$$

For equation (4.2) we have

$$Y_1 = \lim_{T\to\infty} \int_{t_0}^{T} t^{-2}e^{-3t}\bigg(\int_t^T e^r\,dr\bigg)^3\,dt = \infty.$$

Hence, by virtue of Lemma 2.1 we get $J_1 = \infty$. Moreover, we have

$$J_2 = \int_{t_0}^{\infty} e^t\bigg(\int_t^{\infty} r^{-2}e^{-3r}\,dr\bigg)^{1/3}\,dt \geq \int_{t_0}^{\infty}\bigg(\int_t^{\infty} r^{-2}\,dr\bigg)^{1/3}\,dt = \infty, \tag{4.3}$$

and

$$Y_2 = \int_{t_0}^{\infty} t^{-2}e^{-3t}\bigg(\int_{t_0}^{t} e^r\,dr\bigg)^3\,dt \leq \int_{t_0}^{\infty} t^{-2}\,dt < \infty. \tag{4.4}$$

Thus, for equation (4.2) the case (C_5) holds and so, in view of Theorem 3.1, we obtain $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\ell} \neq \varnothing$ and $\mathbb{M}^- = \mathbb{M}^-_{0,0} \neq \varnothing$.

Example 4.3. Consider the half-linear equation for $t \geq t_0 > 0$,

$$\big(t^{2/3}e^t|x'|^{1/3}\operatorname{sgn} x'\big)' - e^t|x|^{1/3}\operatorname{sgn} x = 0. \tag{4.5}$$

For equation (4.5) we have $J_2 = \infty$. Hence, by virtue of Lemma 2.1 we get $Y_2 = \infty$. Using (4.3) and (4.4) we get $J_1 < \infty$ and $Y_1 = \infty$. Then, for equation (4.5) the case (C_8) holds and so, in view of Theorem 3.1 we obtain $\mathbb{M}^+ = \mathbb{M}^+_{\ell,\infty} \neq \varnothing$ and $\mathbb{M}^- = \mathbb{M}^-_{0,0} \neq \varnothing$. Observe that equation (4.5) is the reciprocal equation to (4.2). Hence, the classification of its solutions can be obtained also using the results in Example 4.2 and the Reciprocity principle.

Example 4.4. Consider the half-linear equations for $t \geq t_0 > 1$,

$$\big(|x'|^{1/2}\operatorname{sgn} x'\big)' - t^{-3/2}(\log t)^{-2/3}|x|^{1/2}\operatorname{sgn} x = 0 \tag{4.6}$$

and

$$\big(t^3(\log t)^{4/3}(x')^2\operatorname{sgn} x'\big)' - x^2\operatorname{sgn} x = 0. \tag{4.7}$$

A standard calculation gives for equation (4.6) that $J_2 < \infty$, $Y_2 = \infty$ and $Y_1 = \infty$. Thus, Lemma 2.1 yields $J_1 = \infty$ and the case (C_7) holds. Applying Theorem 3.1 we obtain for equation (4.6) $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \neq \varnothing$ and $\mathbb{M}^- = \mathbb{M}^-_{\ell,0} \neq \varnothing$.

Now, consider equation (4.7). Since this equation is the reciprocal equation to (4.6), for equation (4.7) we have $J_1 = J_2 = Y_2 = \infty$ and $Y_1 < \infty$. Thus, for equation (4.7) the case (C_6) holds and by Theorem 3.1 we get $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \neq \varnothing$ and $\mathbb{M}^- = \mathbb{M}^-_{0,\ell} \neq \varnothing$.

Some applications of Theorem 3.1 to the nonlinear differential equation

$$\big(a(t)|x'|^{\alpha}\operatorname{sgn} x'\big)' - \widetilde{b}(t)F(x) = 0,$$

where the weight $\widetilde{b}$ has indefinite sign and F is a continuous function on $\mathbb{R}$ such that $uF(u) > 0$ for $u \neq 0$, can be found in [9, 10].

References

[1] M. Cecchi, Z. Došlá and M. Marini, On the dynamics of the generalized Emden-Fowler equation. *Georgian Math. J.* **7** (2000), no. 2, 269–282.

[2] M. Cecchi, Z. Došlá and M. Marini, On nonoscillatory solutions of differential equations with p-Laplacian. *Adv. Math. Sci. Appl.* **11** (2001), no. 1, 419–436.

[3] M. Cecchi, Z. Došlá and M. Marini, Integral conditions for nonoscillation of second order nonlinear differential equations. *Nonlinear Anal.* **64** (2006), no. 6, 1278–1289.

[4] M. Cecchi, Z. Došlá, I. Kiguradze and M. Marini, On nonnegative solutions of singular boundary-value problems for Emden–Fowler-type differential systems. *Differential Integral Equations* **20** (2007), no. 10, 1081–1106.

[5] M. Cecchi, M. Marini and G. Villari, On the monotonicity property for a certain class of second order differential equations. *J. Differential Equations* **82** (1989), no. 1, 15–27.

[6] T. A. Chanturia, On singular solutions of nonlinear systems of ordinary differential equations. *Differential equations (Colloq., Keszthely, 1974)*, pp. 107–119. Colloq. Math. Soc. János Bolyai, Vol. 15, *North-Holland, Amsterdam*, 1977.

[7] T. A. Chanturia, On monotone solutions of a system of nonlinear differential equations. **37** (1980), no. 1, 59–70.

[8] Z. Došlá and M. Marini, Second order nonlinear differential and difference equations. *Asymptotic Theory with Applications* (in preparation).

[9] Z. Došlá, M. Marini and S. Matucci, A Dirichlet problem on the half-line for nonlinear equations with indefinite weight. *Ann. Mat. Pura Appl. (4)* **196** (2017), no. 1, 51–64.

[10] Z. Došlá, M. Marini and S. Matucci, Global Kneser solutions to nonlinear equations with indefinite weight. *Discrete Contin. Dyn. Syst. Ser. B* **23** (2018), no. 8, 3297–3308.

[11] Z. Došlá and I. Vrkoč, On an extension of the Fubini theorem and its applications in ODEs. *Nonlinear Anal.* **57** (2004), no. 4, 531–548.

[12] O. Došlý and P. Řehák, *Half-Linear Differential Equations.* North-Holland Mathematics Studies, 202. Elsevier Science B.V., Amsterdam, 2005.

[13] Á. Elbert, A half-linear second order differential equation. *Qualitative theory of differential equations, Vol. I, II (Szeged, 1979)*, pp. 153–180, Colloq. Math. Soc. János Bolyai, 30, *North-Holland, Amsterdam–New York*, 1981.

[14] H. Hoshino, R. Imabayashi, T. Kusano and T. Tanigawa, On second-order half-linear oscillations. *Adv. Math. Sci. Appl.* **8** (1998), no. 1, 199–216.

[15] J. Jaroš, T. Kusano and T. Tanigawa, Nonoscillation theory for second order half-linear differential equations in the framework of regular variation. *Results Math.* **43** (2003), no. 1-2, 129–149.

[16] I. T. Kiguradze, The asymptotic behavior of the solutions of a nonlinear differential equation of Emden-Fowler type. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **29** (1965), 965–986.

[17] I. T. Kiguradze and T. A. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations.* Translated from the 1985 Russian original. Mathematics and its Applications (Soviet Series), 89. Kluwer Academic Publishers Group, Dordrecht, 1993.

[18] I. T. Kiguradze and G. G. Kvinikadze, On strongly increasing solutions of nonlinear ordinary differential equations. *Ann. Mat. Pura Appl. (4)* **130** (1982), 67–87.

[19] T. Kusano, J. Manojlović and T. Tanigawa, Existence of regularly varying solutions with nonzero indices of half-linear differential equations with retarded arguments. *Comput. Math. Appl.* **59** (2010), no. 1, 411–425.

[20] T. Kusano and J. V. Manojlović, Precise asymptotic behavior of regularly varying solutions of second order half-linear differential equations. *Electron. J. Qual. Theory Differ. Equ.* **2016**, Paper No. 62, 24 pp.

[21] T. Kusano and N. Yoshida, Nonoscillation theorems for a class of quasilinear differential equations of second order. *J. Math. Anal. Appl.* **189** (1995), no. 1, 115–127.

[22] G. G. Kvinikadze, Rapidly increasing solutions of systems of nonlinear ordinary differential equations. (Russian) *Tbiliss. Gos. Univ. Inst. Prikl. Mat. Trudy* **37** (1990), 41–49.

[23] V. Marić, *Regular Variation and Differential Equations.* Lecture Notes in Mathematics, 1726. Springer-Verlag, Berlin, 2000.

[24] M. Marini and P. Zezza, On the asymptotic behavior of the solutions of a class of second-order linear differential equations. *J. Differential Equations* **28** (1978), no. 1, 1–17.

[25] J. D. Mirzov, On some analogs of Sturm's and Kneser's theorems for nonlinear systems. *J. Math. Anal. Appl.* **53** (1976), no. 2, 418–425.

[26] J. D. Mirzov, *Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations.* Folia Facultatis Scientiarium Naturalium Universitatis Masarykianae Brunensis. Mathematica, 14. Masaryk University, Brno, 2004.

[27] M. Mizukami, M. Naito and H. Usami, Asymptotic behavior of solutions of a class of second order quasilinear ordinary differential equations. *Hiroshima Math. J.* **32** (2002), no. 1, 51–78.

[28] V. A. Rabtsevich, On rapidly increasing solutions of arbitrary-order Emden–Fowler systems. (Russian) *Differ. Uravn.* **34** (1998), no. 2, 222–227; translation in *Differential Equations* **34** (1998), no. 2, 226–230.

[29] G. Sansone, *Equazioni Differenziali Nel Campo Reale.* Vol. II Zanichelli Ed. Bologna, 1941.

[30] T. Tanigawa, Existence and asymptotic behavior of positive solutions of second order quasilinear differential equations. *Adv. Math. Sci. Appl.* **9** (1999), no. 2, 907–938.