A Condition for the Solvability of the Control Problem of Asynchronous Spectrum of Linear Almost Periodic Systems with the Diagonal Averaging of Coefficient Matrix

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A large number of works are devoted to the study of various questions of control theory for ordinary periodic differential systems (see, for example, [6, 11, 12] and others). For almost periodic control systems, such studies are significantly complicated. In this direction, we can note the results of [5, 9, 10], a characteristic feature of which is the consideration of the so-called regular case, when a priori it is assumed that the frequencies of the system itself and its solutions coincide.

At the same time, as shown by J. Kurzweil and O. Vejvoda [7], the system of ordinary differential almost periodic equations can admit such solutions that the intersection of the frequency modules of the solution and the system is trivial. This result allows us to assume that there exist systems with a very difference spectrum of frequencies, including asynchronous.

In [1], the control problem of the asynchronous spectrum for periodic systems was first formulated. A series of conditions for its solvability are given in the monograph [2, Ch. III]. Similar questions for quasiperiodic systems were studied in [3]. The control problem of the asynchronous spectrum of linear almost periodic systems was formulated in [4] and the case of trivial mean value of the coefficient matrix is considered.

Now we study the solvability of the control problem of the asynchronous spectrum of linear almost periodic systems for which the mean value of the coefficient matrix is diagonal.

Let $f(t)$ be a real almost periodic (Borh) function [2]. The mean value of an almost periodic function $f(t)$ is determined by the equality

$$\widehat{f} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\,dt.$$

The modulus (frequency modulus) $\mathrm{Mod}(f)$ of an almost periodic function $f(t)$ is the smallest additive group of real numbers that contains all the Fourier exponents (frequencies) of this function. Let be $g(t,x)$ a vector-function that is almost periodic in uniformly relative to some compact set. J. Kurzweil and O Vejvoda proved that the system of ordinary differential equations

$$\dot{x} = g(t,x)$$

can have an almost periodic solution $x(t)$ such that the intersection of the frequency modules of the solution and the right-hand side is trivial, i.e.

$$\mathrm{Mod}(x) \cap \mathrm{Mod}(g) = \{0\}.$$

In what follows, such almost periodic solutions will be called strongly irregular.

Consider the linear non-stationary control system

$$\dot{x} = A(t)x + Bu, \ \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \ \ n \geq 2, \tag{1}$$

where x is the phase vector of the system, u is the input, B is the constant $n \times n$-matrix under control, $A(t)$ is a continuous almost periodic matrix with a modulus of frequencies $\mathrm{Mod}(A)$. Suppose that the control is specified in the form of a feedback linear in the phase variables

$$u = U(t)x$$

with a continuous almost periodic $n \times n$-matrix $U(t)$ (feedback coefficient), the frequency modulus of which is contained in the frequency modulus of the coefficient matrix, i.e.,

$$\mathrm{Mod}(U) \subseteq \mathrm{Mod}(A).$$

It is required to obtain conditions on the right-hand side of system (1) such that for any choice of the feedback coefficient from the indicated admissible set, the closed-loop system

$$\dot{x} = (A(t) + BU(t))x,$$

does not have a strongly irregular almost periodic solution, the frequency spectrum of which contains a given subset (target set). In other words, for system (1) it is necessary to find the conditions for the unsolvability of the problem of control of the asynchronous spectrum.

We suppose that the coefficient matrix has a diagonal average value, i.e.,

$$\widehat{A} = \mathrm{diag}(\widehat{a}_{11}, \dots, \widehat{a}_{nn}), \quad \widehat{a}_{11}^2 + \cdots \widehat{a}_{nn}^2 \neq 0. \tag{2}$$

Consider the case when the matrix under control is singular, i.e.

$$\mathrm{rank}\, B = r \ \ (1 \leq r < n), \tag{3}$$

moreover, its first rows are zero. Let us denote by $B_{r,n}$ the matrix composed of the remaining rows of the matrix B. The rank of the matrix $B_{r,n}$ is also equal to r. Taking into account the representation (3) of the matrix B, the matrix of coefficients $A(t)$ is divided into four blocks of the corresponding dimensions (indicated by the subscripts):

$$A(t) = \begin{pmatrix} A_{d,d}^{(11)}(t) & A_{d,r}^{(12)}(t) \\ A_{r,d}^{(21)}(t) & A_{r,r}^{(22)}(t) \end{pmatrix}.$$

Taking into account condition (2), we write the average value of the coefficient matrix in the form

$$\widehat{A} = \begin{pmatrix} \widehat{A}_{d,d}^{(11)} & 0 \\ 0 & \widehat{A}_{r,r}^{(22)} \end{pmatrix},$$

where $\widehat{A}_{d,d}^{(11)} = \mathrm{diag}(\widehat{a}_{11}, \dots, \widehat{a}_{dd})$, $\widehat{A}_{r,r}^{(22)} = \mathrm{diag}(\widehat{a}_{d+1\,d+1}, \dots, \widehat{a}_{nn})$. Then the oscillating part of the coefficient matrix is also represented in the following block form:

$$\widetilde{A}(t) = A(t) - \widehat{A} = \begin{pmatrix} \widetilde{A}_{d,d}^{(11)}(t) & A_{d,r}^{(12)}(t) \\ A_{r,d}^{(21)}(t) & \tilde{A}_{r,r}^{(22)} \end{pmatrix}.$$

Denote by q the column rank of a rectangular $d \times n$-matrix $(\widetilde{A}_{d,d}^{(11)} \ A_{d,r}^{(12)})$.

The following theorem holds.

Theorem. *Let conditions* (3) *and the equality* $q = n$ *hold. Then the problem of control of the asynchronous spectrum of system* (1) *has no solution.*

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