Boundary Value Problems for Systems of Nonsingular Integral-Differential Equations of Fredholm Type with Degenerate Kernel

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We investigate the problem of finding solutions [1]

$$y(t)\in\mathbb{D}^2[a;b],\ \ y'(t)\in\mathbb{L}^2[a;b]$$

of the linear Noetherian ($n\neq v$) boundary value problem for a system of linear integral-differential equations of Fredholm type with degenerate kernel

$$A(t)y'(t)=B(t)y(t)+\Phi(t)\int\limits_a^b F(y(s),y'(s),s)\,ds+f(t),\ \ \ell y(\,\cdot\,)=\alpha,\ \ \alpha\in\mathbb{R}^p. \tag{1}$$

We seek a solution of the Noetherian boundary value problem (1) in a small neighborhood of the solution

$$y_0(t)\in\mathbb{D}^2[a;b],\ \ y_0'(t)\in\mathbb{L}^2[a;b]$$

of the generating problem

$$A(t)y_0'(t)=B(t)y_0(t)+f(t),\ \ \ell y_0(\,\cdot\,)=\alpha. \tag{2}$$

Here

$$A(t),B(t)\in\mathbb{L}^2_{m\times n}[a;b]:=\mathbb{L}^2[a;b]\otimes\mathbb{R}^{m\times n},\ \ \Phi(t)\in\mathbb{L}^2_{m\times q}[a;b],\ \ f(t)\in\mathbb{L}^2[a;b].$$

We assume that the matrix $A(t)$ is, generally speaking, rectangular: $m\neq n$. It can be square, but singular. Assume that the function $F(y(t),y'(t),t)$ is linear with respect to unknown $y(t)$ in a small neighborhood of the generating solutions and with respect to the derivative $y'(t)$ in a small neighborhood of the function $y_0'(t)$. In addition, we assume that the function $F(y(t),y'(t),t)$ is continuous in the independent variable t on the segment $[a,b]$;

$$\ell y(\,\cdot\,):\mathbb{D}^2[a;b]\to\mathbb{R}^p$$

is a linear bounded vector functional defined on a space $\mathbb{D}^2[a;b]$. The problem of finding solutions of the boundary value problem (1) in case $A(t)=I_n$ was solved by A. M. Samoilenko and A. A. Boichuk [7]. Thus, the boundary value problem (1) is a generalization of the problem solved by A. M. Samoilenko and A. A. Boichuk.

We investigate the problem of finding solutions of the linear Noetherian boundary value problem (2) in the paper [9]. Under the condition

$$P_{A^*}(t)=0,\ \ \operatorname{rank}A(t):=\sigma_0=m\leq n \tag{3}$$

we arrive at the problem of construction of solutions of the linear differential-algebraic system [9]

$$z' = A^{+}(t)B(t)z + \mathfrak{F}_0(t, \nu_0(t)); \tag{4}$$

here,

$$\mathfrak{F}_0(t, \nu_0(t)) := A^{+}(t)f(t) + P_{A_{\rho_0}}(t)\nu_0(t),$$

$A^{+}(t)$ is a pseudoinverse (by Moore–Penrose) matrix [1], $\nu_0(t) \in \mathbb{L}^2[a;b]$ is an arbitrary vector function. In addition, $P_{A^*(t)}$ is a matrix-orthoprojector [1]:

$$P_{A^*}(t) : \mathbb{R}^m \to \mathbb{N}(A^*(t)),$$

$P_{A_{\rho_0}}(t)$ is an $(n \times \rho_0)$-matrix composed of ρ_0 linearly independent columns of the $(n \times n)$-matrix-orthoprojector:

$$P_A(t) : \mathbb{R}^n \to \mathbb{N}(A(t)).$$

By analogy with the classification of pulse boundary-value problems [1–3,8] we say in the (3), that, system of linear integral-differential equations is nonsingular. Denote by $X(t)$ normal fundamental matrix

$$X'(t) = A^{+}(t)B(t)X(t), \quad X(a) = I_n.$$

Substituting the general solution of the system of linear integral-differential equations (3) into the boundary condition (1), we arrive at the linear algebraic equation

$$Q\,c = \ell K[f(s)](\,\cdot\,). \tag{5}$$

In the critical case

$$P_{Q^*} \neq 0, \;\; Q := \ell X(\,\cdot\,) \in \mathbb{R}^{p\times n},$$

equation (5) is solvable iff

$$P_{Q_d^*}\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} = 0. \tag{6}$$

Here, $P_{Q_d^*}$ is an $(d \times p)$-matrix composed of d linearly independent rows of the $(p \times p)$-matrix-orthoprojector:

$$P_{Q^*} : \mathbb{R}^p \to \mathbb{N}(Q^*).$$

Thus, the following lemma is proved [9].

Lemma 1. *In the critical case $P_{Q^*} \neq 0$, the nonsingular differential-algebraic boundary value problem* (2) *is solvable iff* (6) *holds. In the critical case, the nonsingular differential-algebraic boundary value problem* (2) *has a solution of the form*

$$y_0(t, c_r) = X_r(t)c_r + G[f(s);\alpha](t), \quad X_r(t) := X(t)P_{Q_r}, \;\; c_r \in \mathbb{R}^r,$$

which depends on the arbitrary vector-function $\nu_0(t) \in \mathbb{L}^2[a;b]$. Here, P_{Q_r} is an $(p \times r)$-matrix composed of r linearly independent columns of the $(p \times p)$-matrix-orthoprojector: $P_Q : \mathbb{R}^p \to \mathbb{N}(Q)$;

$$G[f(s);\alpha](t) := X(t)Q^{+}\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} + K[f(s)](t)$$

is the generalized Green operator of the linear integral-differential problem (1)*;*

$$K\big[f(s)\big](t) := X(t)\int\limits_a^t X^{-1}(s)\mathfrak{F}_0(s, \nu_0(s))\,ds$$

is the generalized Green operator of the Cauchy problem for the integral-differential system (3)*.*

In problem (1) we perform the substitution $y(t)=y_0(t,c_r)+x(t)$. For

$$x(t)\in\mathbb{D}^2[a;b],\ \ x'(t)\in\mathbb{L}^2[a;b],\ \ x(t,u,v)=X_0(t)u+\Psi(t)v$$

we obtain the problem

$$A(t)x'(t)=B(t)x(t)+\Phi(t)\int\limits_a^b F(y(s),y'(s),s)\,ds,\ \ \ell x(\,\cdot\,)=0.$$

Here,

$$v:=\int\limits_a^b F(y(s),y'(s),s)\,ds\in\mathbb{R}^q,\ \ u\in\mathbb{R}^n,\ \ \Psi(t):=K[\Phi(s)](t)\in\mathbb{D}^2_{n\times q}[a;b].$$

Denote the matrix

$$\check{Q}:=[Q;R]\in\mathbb{R}^{p\times(q+n)},\ \ R:=\ell\,\Psi(\,\cdot\,)\in\mathbb{R}^{p\times q}$$

and $P_\rho\in\mathbb{R}^{(q+n)\times\rho}$ composed of ρ linearly independent columns of the matrix-orthoprojector $P_{\check{Q}}$:

$$P_{\check{Q}}:\mathbb{R}^{q+n}\to\mathbb{N}(\check{Q}).$$

Substituting the general solution of the system of the linear integral-differential system (1) into the boundary condition (1), we arrive at the linear algebraic equation

$$Q\,v+R\,u=0,\ \ R:=\ell\,\Psi(\,\cdot\,)\in\mathbb{R}^{p\times q}.$$

Using the continuous differentiability of the function $F(y(t),y'(t),t)$ with respect to unknown $y(t)$ in a small neighborhood of the generating solutions and with respect to the derivative $y'(t)$ in a small neighborhood of the function $y_0'(t)$, we expand this function

$$F(y(t),y'(t),t)=A_1(t)y(t)+A_2(t)y'(t),\ \ A_1(t):=F_y'(y(t),y'(t),t),\ \ A_2(t):=F_{y'}'(y(t),y'(t),t).$$

Applying Lemma 1 to the boundary value problem (1), we obtain equation

$$\mathcal{B}_0\,c_\rho+\psi(c_r)=0,\quad \psi(c_r):=-\int\limits_a^b\big[A_1(t)y_0(t,c_r)+A_2(t)y_0'(t,c_r)\big]\,dt,\tag{7}$$

where

$$\mathcal{B}_0:=\int\limits_a^b\Big\{A_1(t)[X(t)P_1+\Psi(t)P_2]+A_2(t)\big[X'(t)P_1+\Psi'(t)P_2\big]\Big\}\,dt-P_2,$$

In the critical case $P_{\mathcal{B}_0^*}\neq0$, equation (6) is solvable iff

$$P_{\mathcal{B}_0^*}\psi(c_r)=0.\tag{8}$$

Here,

$$P_{\mathcal{B}_0^*}:\mathbb{R}^q\to\mathbb{N}(\mathcal{B}_0^*),\quad P_{\mathcal{B}_0}:\mathbb{R}^\rho\to\mathbb{N}(\mathcal{B}_0)$$

is matrices-orthoprojectors. Under condition (8) and only under it nonsingular system of linear integral-differential equations (1) has a solution of the form

$$y(t,c_\mu)=Y_\mu(t)c_\mu+G[\Phi(s);\nu_0(s)](t),\ \ c_\mu\in\mathbb{R}^\mu,$$

which depends on the arbitrary vector-function $\nu_0(t) \in \mathbb{L}^2[a;b]$. Here,

$$G[\Phi(s);\nu_0(s)](t) := G[f(s);\alpha](t) - \mathcal{B}_0^+ \int\limits_a^b \Big[A_1(t)G[f(s);\alpha](t) + A_2(t)G'[f(s);\alpha](t)\Big]\,dt$$

is the generalized Green operator of the linear integral-differential problem (1), where $Y_\mu(t)$ is an $(n \times \mu)$-matrix composed of μ linearly independent columns of the matrix:

$$\Big[X_0(t) - \big[X_0(t)P_1 + \Psi(t)P_2\big]\mathcal{B}_0^+\big[A_1(t)X_r(t); A_2(t)X_r'(t)\big]\,dt; \big[X(t)P_1 + \Psi(t)P_2\big]P_{\mathcal{B}_0}\Big].$$

Thus, the following theorem is proved.

Theorem 1. *In the critical case, under condition* (6) *the nonsingular integral-differential boundary value problem* (3) *has a solution of the form*

$$y_0(t,c_r) = X_r(t)c_r + G[f(s);\alpha](t), \quad X_r(t) := X(t)P_{Q_r}, \ \ c_r \in \mathbb{R}^r,$$

which depends on the arbitrary vector-function $\nu_0(t) \in \mathbb{L}^2[a;b]$. *Under condition* (8) *and only under it the general solution of the nonsingular integral-differential boundary value problem* (1)

$$y(t,c_\mu) = Y_\mu(t)c_\mu + G[\Phi(s);\nu_0(s)](t), \ \ c_\mu \in \mathbb{R}^\mu$$

is determined by the generalized Green operator of the nonsingular integral-differential boundary value problem (1).

The proposed scheme of studies of the nonsingular integral-differential boundary value problem (1) can be transferred analogously to [1, 5, 6] onto nonlinear nonsingular integral-differential boundary value problem. On the other hand, in the case of nonsolvability, the nonsingular integral-differential boundary value problems can be regularized analogously [4, 10].

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