Asymptotic Properties of $P_\omega(Y_0, Y_1, \pm\infty)$-Solutions of Second Order Differential Equations with the Product of Different Classes of Nonlinearities

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We consider the following differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'). \tag{1}$$

In this equation $\alpha_0 \in \{-1; 1\}$, functions $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$ and $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i \in \{0, 1\})$ are continuous, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[$ or the interval $]Y_i, y_i^0]$. If $Y_i = +\infty$ $(Y_i = -\infty)$, we put $y_i^0 > 0$ $(y_i^0 < 0)$.

We also suppose that function φ_1 is a regularly varying as $y \to Y_1$ function of index σ_1 [10, p. 10-15], function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y) \ne 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi_0(y) \in \{0, +\infty\}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{2}$$

It follows from the above conditions (2) that the function φ_0 and its derivative of the first order are rapidly varying functions as the argument tends to Y_0 [10, p. 15]. Thus, the investigated differential equation contains the product of regularly and rapidly varying nonlinearities in its right-hand side.

The equations of the form (1) often appear in practice, for example, in the theory of burning, when we consider the electrostatic potential in a spherical volume of plasma of products of burning. First important results in this direction have been obtained in the works by V. M. Evtukhov for the equation of the investigated type in the case when $\varphi_0(y) = |y|^\sigma$ and $\varphi_1(y') = |y'|^\lambda$.

For the equation of the form (1), both functions φ_0 and φ_1 of which are regularly varying functions of orders σ_0 and σ_1 correspondingly $(\sigma_0 + \sigma_1 \ne 1)$ as their arguments tend to zero or to infinity, the asymptotic behavior of some class of solutions have been investigated in the works by M. O. Belozerova [5].

During investigations of distribution of electrostatic potential in a cylindrical plasma volume of combustion products, differential equation of the investigated type arises, in which $\varphi_0(y) = \exp(\sigma y)$, $\varphi_1(y') = |y'|^\lambda$, $\alpha_0 \in \{-1, 1\}$, $\sigma, \lambda \in \mathbb{R}$, $\sigma \ne 0$, function $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$ is a continuously differentiable function. Under some restrictions on the function $p(t)$ certain results for the asymptotic behavior of all regular solutions of this equation have been obtained in works by V. M. Evtukhov and N. G. Dric (see [7], for example).

Equations, that contain in their right-hand side the product of functions $\varphi_0(y)$ and $\varphi_1(y')$, the first one of which is a rapidly varying function as $y \to Y_0$ $(y \in \Delta_{Y_0})$, and the second one is a regularly varying function as $y' \to Y_1$ $(y' \in \Delta_{Y_1})$, in general case have not been investigated before. Thus, equation (1) plays an important role in the development of a qualitative theory of differential equations.

The main aim of the article is the investigation of conditions for the existence of following class of solutions of equation (1).

Definition 1. A solution y of equation (1), defined on the interval $[t_0,\omega[\subset[a,\omega[$, is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution ($-\infty\le\lambda_0\le+\infty$) if the following conditions take place

$$y^{(i)}:[t_0,\omega[\to\Delta_{Y_i},\quad \lim_{t\uparrow\omega}y^{(i)}(t)=Y_i\ \ (i=0,1),\quad \lim_{t\uparrow\omega}\frac{(y'(t))^2}{y''(t)y(t)}=\lambda_0.$$

This class of solutions was defined in the work of V. M. Evtukhov [3] for the n-th order differential equations of Emden–Fowler type and was concretized for the second-order equation. Due to the asymptotic properties of functions in the class of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions [4], every such solution belongs to one of four non-intersecting sets according to the value of $\lambda_0:\lambda_0\in\mathbb{R}\setminus\{0,1\}$, $\lambda_0=0$, $\lambda_0=1$, $\lambda_0=\pm\infty$. In this article we consider the case $\lambda_0=\pm\infty$ of such solutions, every $P_\omega(Y_0,Y_1,\pm\infty)$-solution and its derivative satisfy the following limit relations

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'(t)}{y(t)}=1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}=0. \tag{3}$$

This class of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions for equations of the form (1) is one of the most difficult to study due to the fact that the second-order derivative is not explicitly expressed through the first-order derivative. From (3) it means that the derivative of the first order of each such solution is a slowly varying function as $t\uparrow\omega$.

From conditions (2) it also follows that the function φ_0 and its first-order derivative belong to the class $\Gamma_{Y_0}(Z_0)$, that was introduced in the works of V. M. Evtukhov and A. G. Chernikova [6] as a generalization of the class Γ (L. Khan, see, for example, [1, p. 75]). The properties of the class $\Gamma_{Y_0}(Z_0)$ were used to get our results.

To formulate the main results, we introduce the following definitions.

Definition 2. Let $Y\in\{0,\infty\}$, Δ_Y is some one-sided neighborhood of Y. Continuous-differentiable function $L:\Delta_Y\to]0,+\infty[$ is called [9, p. 2-3] a normalized slowly varying function as $z\to Y$ ($z\in\Delta_Y$) if the next statement is valid

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}}\frac{yL'(y)}{L(y)}=0.$$

Definition 3. We say that a slowly varying as $z\to Y$ ($z\in\Delta_Y$) function $\theta:\Delta_Y\to]0,+\infty[$ satisfies the condition S as $z\to Y$, if for any continuous differentiable normalized slowly varying as $z\to Y$ ($z\in\Delta_Y$) function $L:\Delta_{Y_i}\to]0,+\infty[$ the next relation is valid

$$\theta(zL(z))=\theta(z)(1+o(1))\ \text{ as }\ z\to Y\ \ (z\in\Delta_Y).$$

Condition S is satisfied, for example, for such functions as $\ln|y|$, $|\ln|y||^\mu$ ($\mu\in\mathbb{R}$), $\ln\ln|y|$.

The following theorem is obtained in our previous work [2] and contains a necessary conditions for the existence of the $P_\omega(Y_0,Y_1,\pm\infty)$-solution of equation (1).

Theorem 1 ([2])**.** *Let for equation* (1) $\sigma_1\neq1$, *the function* $\varphi_1(y')|y'|^{-\sigma_1}$ *satisfy the condition* S *as* $y'\to Y_1$ ($y'\in\Delta_{Y_1}$). *Then each* $P_\omega(Y_0,Y_1,\pm\infty)$*-solution of the differential equation* (1) *can be represented as*

$$y(t)=\pi_\omega(t)L(t),$$

where $L:[t_0,\omega[\to\mathbb{R}$ *is twice continuously differentiable on* Δ_{Y_0} *and satisfies the next conditions*

$$y_0^0\pi_\omega(t)L(t)>0,\quad L'(t)\neq0\ \ \text{as}\ \ t\in[t_1,\omega[\ \ (t_0\le t_1<\omega),$$

$$\lim_{t\uparrow\omega}L(t)\in\{0;\pm\infty\},\quad \lim_{t\uparrow\omega}\pi_\omega(t)L(t)=Y_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)L'(t)}{L(t)}=0. \tag{4}$$

Thus, in the case of the existence of a finite or infinite limit

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)L''(t)}{L'(t)}, \tag{5}$$

the following relations take place

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)L''(t)}{L'(t)}=-1,\quad \alpha_0L'(t)>0\ \ as\ \ t\in[t_1,\omega[\ \ (t_0\le t_1<\omega), \tag{6}$$
$$p(t)=\frac{\alpha_0L'(t)}{\varphi_1(L(t))\varphi_0(\pi_\omega(t)L(t))}\,[1+o(1)]\ \ as\ \ t\uparrow\omega.$$

Let us introduce the following definition.

Definition 4. We say that the condition N is satisfied for equation (1) if for some continuously differentiable function $L(t):[t_0,\omega[\,\to\mathbb{R}$ ($t_0\in[a,\omega[$), which satisfies conditions (4)–(6), the following representation takes place

$$p(t)=\frac{\alpha_0L'(t)}{\varphi_1(L(t))\varphi_0(\pi_\omega(t)L(t))}\,[1+r(t)],$$

where $r(t):[t_0,\omega[\,\to\,]-1,+\infty[$ is a continuous function that tends to zero as $t\uparrow\omega$.

For equation (1), in previous works [2] the necessary and sufficient conditions for the existence of the investigated class of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions were established in case of the existence of some infinite limit. In this work we establish sufficient conditions for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of equation (1) in case this limit equals nonzero real number. We also have found the asymptotic representations of such solutions and its first order derivatives as $t\uparrow\omega$ and indicated the number of such solutions.

To formulate the sufficient conditions for the existence of the $P_\omega(Y_0,Y_1,\pm\infty)$-solution of equation (1), let us introduce some notations:

$$\mu_0=\operatorname{sign}\varphi_0'(y),\quad \theta_1(y')=\varphi_1(y')|y'|^{-\sigma_1},$$
$$H(t)=\frac{L^2(t)\varphi_0'(\pi_\omega(t)L(t))}{L'(t)\varphi_0(\pi_\omega(t)L(t))}\,,\quad q_1(t)=\left.\frac{\big(\frac{\varphi_0'(y)}{\varphi_0(y)}\big)'}{\big(\frac{\varphi_0'(y)}{\varphi_0(y)}\big)^2}\right|_{y=\pi_\omega(t)L(t)},$$
$$e_1(t)=1+\frac{\pi_\omega(t)L'(t)}{L(t)}\,,\quad e_2(t)=2+\frac{\pi_\omega(t)L''(t)}{L'(t)}\,.$$

For these functions the following statements are fulfilled:

1)
$$\lim_{t\uparrow\omega}e_1(t)=\lim_{t\uparrow\omega}e_2(t)=1,\quad \lim_{t\uparrow\omega}H(t)=\pm\infty,\quad \lim_{t\uparrow\omega}q_1(t)=0,$$

2) If the limit
$$\lim_{t\uparrow\omega}\frac{L(t)}{L'(t)}\cdot\frac{H'(t)}{|H(t)|^{\frac32}}$$
exists, then
$$\lim_{t\uparrow\omega}\frac{L(t)}{L'(t)}\cdot\frac{H'(t)}{|H(t)|^{\frac32}}=0. \tag{7}$$

The sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1) in case

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)L'(t)}{L(t)}\, |H(t)|^{\frac{1}{2}} = \pm\infty,$$

were found in [2].

In this work we suppose that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)L'(t)}{L(t)}\, |H(t)|^{\frac{1}{2}} = \gamma, \quad 0 < |\gamma| < +\infty. \tag{8}$$

The sufficient conditions for this case are formulated in the following theorem.

Theorem 2. *Let for equation* (1) $\sigma_1 \neq 1$*, the function* $\varphi_1(y')|y'|^{-\sigma_1}$ *satisfy the condition* S *as* $y' \to Y_1$ $(y' \in \Delta_{Y_1})$*, the conditions* N, (7) *and* (8) *hold. Then*

- *in case* $\alpha_0\mu_0 > 0$*, the differential equation* (1) *has a one-parametric family of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions;*
- *in case* $\alpha_0\mu_0 < 0$ *and* $y_0^0\alpha_0\gamma\pi_\omega(t) < 0$*, the differential equation* (1) *has a two-parametric family of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions;*
- *in case* $\alpha_0\mu_0 < 0$ *and* $y_0^0\alpha_0\gamma\pi_\omega(t) > 0$*, the differential equation* (1) *has at least one of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions.*

For each of such solutions the following asymptotic representations take place as $t \uparrow \omega$*,*

$$y(t) = \pi_\omega(t)\cdot L(t) + \frac{\varphi_0(\pi_\omega(t)L(t))}{\varphi_0'(\pi_\omega(t)L(t))}\cdot o(1),$$
$$y'(t) = \big[L(t) + \pi_\omega(t)\cdot L'(t)\big]\cdot\big[1 + |H(t)|^{-\frac{1}{2}}\cdot o(1)\big].$$

For the equation under the investigation the question of the active existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions, that have the obtained asymptotic representations, has been reduced to the question of the existence of infinitely small as arguments tend to ω solutions of the corresponding, equivalent to the investigated equation, systems of non-autonomous quasi-linear differential equations that admit applications of the known results from the works of V. M. Evtukhov and A. M. Samoilenko [8].

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