On Solvability of Two-Point Boundary Value Problems with Separating Boundary Conditions for Linear Ordinary Differential Equations and Totally Positive Kernels

Eugene Bravyi

Perm National Research Polytechnic University, Perm, Russia
E-mail: bravyi@perm.ru

Sulkhan Mukhigulashvili

Brno University of Technology, Brno, Czech Republic
E-mail: mukhigulashvili@fbm.vutbr.cz

We consider two-point boundary value problems for higher order linear ordinary differential equations

$$\begin{cases} x^{(n)}(t)+p^+(t)x(t)-p^-(t)x(t)=f(t), & t\in[a,b],\\ x^{(i)}(0)=0, & i=0,\dots,m-1,\\ x^{(j)}(1)=0, & j=0,\dots,n-m-1, \end{cases} \tag{1}$$

where n, m are positive integer, $n>m$; $f\in\mathbf{L}[a,b]$, $p=p^+-p^-\in[a,b]$,

$$p^+(t)=\begin{cases} p(t) & \text{if } p(t)\ge 0,\\ 0 & \text{if } p(t)<0, \end{cases}\quad p^-(t)=\begin{cases} -p(t) & \text{if } p(t)<0,\\ 0 & \text{if } p(t)\ge 0, \end{cases}$$

$\mathbf{L}[a,b]$ is the space of Lebesgue integrable functions with the standard norm. Together with (1), we will consider some more general problems.

It is a rather common case, when problem (1) has a unique solution for all functions p^+ (or for all functions p^-) with a fixed another function p^- (or p^+). So, our aim is to find some conditions to our boundary value problem (1) to be uniquely solvable for all integrable non-negative coefficients p^+ (or for all nonnegative coefficients p^-).

In this report, we would like to remind about some classical results by F. R. Gantmacher, M. G. Krein, S. Karlin, A. Yu. Levin [1–5]. These results allow us to find required conditions in a very simple way. For higher-order equations, we don't know another proof of these conditions, for example, by means of mathematical analysis only.

A continuous function $G(\,\cdot\,,\,\cdot\,):[a,b]\times[a,b]\to R$ is called a totally positive kernel [3] if all determinants

$$\begin{vmatrix} G(t_1,t_1) & \dots & G(t_1,t_k)\\ \vdots & \ddots & \vdots\\ G(t_k,t_1) & \dots & G(t_k,t_k) \end{vmatrix}$$

are positive for all ordered sets of points $a<t_1<\dots<t_k<b$ for all integer positive numbers k.

It is very hard to check this property directly. Fortunately, Green functions of many boundary value problems for ordinary differential equations possess this property. Now we can formulate a well-known statement on the spectrum of integral operators with totally positive kernels.

Let $G(t,s)$ be a totally positive kernel, $\mathbf{C}[a,b]$ be the space of real continuous functions with the standard norm.

Consider the integral operator $G : \mathbf{C}[a,b] \to \mathbf{C}[a,b]$

$$(Gx)(t) = \int_a^b G(t,s)x(s)\,ds, \quad t \in [a,b]. \tag{2}$$

Theorem 1 (Sturm, Kellogg, Gantmacher, Krein, Karlin, Levin, Stepanov)**.** *The spectrum of the operators G is a subset of the set $[0,\infty)$.*

In the symmetric case, oscillating properties of the spectrum were known to Kellogg and Sturm. F. R. Gantmacher and M. G. Krein [1] showed that kernels could be non-symmetric and proved oscillation properties of the spectrum of many boundary value problems. Here we need only the positivity of the spectrum and we do not mention all remarkable oscillation properties. So, if the kernel $G(t,s)$ is totally positive, then the non-zero spectrum of operator (2) is positive. The next obvious step is only a more general formulation.

Let $G(t,s)$ be a totally positive kernel, $r \in \mathbf{L}[a,b]$, $r(t) \geq 0$, $t \in [a,b]$.

Consider the integral operator $G_r : \mathbf{C}[a,b] \to \mathbf{C}[a,b]$

$$(G_r x)(t) = \int_a^b G(t,s)r(s)x(s)\,ds, \quad t \in [a,b].$$

Theorem 2 (Sturm–Kellogg–Gantmacher–Krein)**.** *The spectrum of the operators G_r is a subset of the set $[0,\infty)$.*

Therefore, in this case all characteristic values λ of the equation

$$x(t) = \lambda \int_a^b G(t,s)r(s)x(s)\,ds, \quad t \in [a,b],$$

are positive.

Now consider two-point boundary value problems for linear higher order ordinary differential equations

$$\begin{cases} (Lx)(t) \equiv x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \cdots + p_n(t)x(t) = f(t), \\ \ell_i x \equiv \displaystyle\sum_{k=0}^{k_i-1} \gamma_{ik} x^{(k)}(a) + x^{(k_i)}(a), \quad i = 1,\dots,m, \\ \ell_i x \equiv \displaystyle\sum_{k=0}^{k_i-1} \gamma_{ik} x^{(k)}(b) + x^{(k_i)}(b), \quad i = m+1,\dots,n, \end{cases} \tag{3}$$

where $p_i \in \mathbf{L}[a,b]$; $f \in \mathbf{L}[a,b]$; n, m, $n > m$, are positive integers; $k_i \in \{0,1,\dots,n-1\}$, $i = 1,\dots,n$. Denote $\ell = \{\ell_1,\dots,\ell_n\}$.

Theorem 3. *If $r \in \mathbf{L}[a,b]$, $r(t) \geq 0$, $t \in [a,b]$, and the Green function $G(t,s)$ of this problem* (3) *is a totally positive kernel, then problem*

$$\begin{cases} (Lx)(t) + r(t)x(t) = f(t), \quad t \in [a,b], \\ \ell x = 0, \end{cases} \tag{4}$$

is uniquely solvable.

Proof. For $f \equiv 0$, (4) is equivalent to the integral equation

$$x(t) = \lambda \int\limits_a^b G(t,s) r(s) x(s)\, ds, \quad t \in [a,b], \tag{5}$$

where $\lambda = -1$. All eigenvalues of (5) are positive, therefore, problem (4) is uniquely solvable. $\square$

Together with the differential operator L, we consider operators L^+ and L^-:

$$\begin{aligned}(L^+x)(t) &\equiv x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \cdots + p_n^+(t)x(t), \quad t \in [a,b],\\ (L^-x)(t) &\equiv x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \cdots - p_n^-(t)x(t), \quad t \in [a,b].\end{aligned}$$

Theorem 4. *Let $G^+(t,s)$ be the Green function of the problem $L^+x = f$, $\ell x = 0$. If $-G^+(t,s)$ is a totally positive kernel, then the problem*

$$\begin{cases}(Lx)(t) = f(t),\\ \ell_i x = 0, & i = 1,\dots,n,\end{cases} \tag{6}$$

is uniquely solvable for all non-negative functions $p_n^- \in \mathbf{L}[a,b]$.

Theorem 5. *Let $G^-(t,s)$ be the Green function of the problem $L^-x = f$, $\ell x = 0$. If $G^-(t,s)$ is a totally positive kernel, then problem* (6) *is uniquely solvable for all non-negative functions $p_n^+ \in \mathbf{L}[a,b]$.*

We say that the differential operator L (or the equation $Lx = 0$) is non-oscillating on the interval $[a,b]$ if every non-trivial solution has no more than $n-1$ zeros in the interval $[a,b]$ taking into account the multiplicity of the zeros. Hartman-Levin's criterion for non-oscillation can be found, for example, in [4].

Theorem 6 (Gantmacher–Krein, see [4, 5])**.** *Let $Lx = 0$ be non-oscillating, $G(t,s)$ the Green function of the problem*

$$\begin{cases}(Lx)(t) = f(t),\\ x^{(i-1)}(a) = 0, & i = 1,\dots,m,\\ x^{(i-1)}(b) = 0, & i = 1,\dots,n-m.\end{cases}$$

Then $(-1)^{n-m}G(t,s)$ is a totally positive kernel.

Let the operator L be non-oscillating. Then L has the Polia–Mammana decomposition

$$Lx = r_0 \frac{d}{dt}\, r_1 \frac{d}{dt} \cdots r_{n-1} \frac{d}{dt}\, r_n,$$

where r_i, $i = 0,\dots,n$, are sufficiently smooth positive functions. Let $G(t,s)$ be the Green function of the uniquely solvable problem

$$\begin{cases}(Lx)(t) = f, & t \in [a,b],\\ \displaystyle\sum_{k=1}^n \alpha_{ik}(D_{k-1})(a) = 0, & i = 1,\dots,m,\\ \displaystyle\sum_{k=1}^n \beta_{ik}(D_{k-1})(b) = 0, & i = 1,\dots,n-m,\end{cases}$$

where $f \in \mathbf{L}[a,b]$; $D_0x = x$, $D_kx = \frac{d}{dt}\,(r_{n-k+1}D_{k-1}x)$, $k = 1,\dots,n$; n, m, $n > m$, positive integers.

Theorem 7 (Kalafaty–Gantmacher–Krein, see [4, 5])**.** *If all m-th order minors of the matrix $||(-1)^k\alpha_{ik}||_{i=1,\dots,m}^{k=1,\dots,n}$ and all $(n-m)$-th order minors of $||\beta_{ik}||_{i=1,\dots,n-m}^{k=1,\dots,n}$ have the same sign, then $(-1)^{n-m}G(t,s)$ is a totally positive kernel.*

Theorem 8 (Levin–Stepanov, see [4,5])**.** *Let $G(t,s)$ be the Green function of the uniquely solvable problem*

$$\begin{cases} (Lx)(t)=f(t), & t\in[a,b],\\ \displaystyle\sum_{k=0}^{k_i-1}\gamma_{ik}x^{(k)}(a)+x^{(k_i)}(a), & i=1,\dots,m,\\ \displaystyle\sum_{k=0}^{k_i-1}\gamma_{ik}x^{(k)}(b)+x^{(k_i)}(b), & i=1,\dots,n-m, \end{cases}$$

$n_k=|\{i:\ k_i\le k, i=1,\dots,n\}|$, $h=2(b-a)$.

If $n_k>k$, $k=0,1,\dots,n-2$, and

$$\sum_{k=1}^{n}h^{k-1}\int\limits_a^b|p_k(t)|\,dt<\frac12\,,\quad \sum_{k=0}^{k_i-1}|\gamma_{ik}|h^{k_i-k}<\frac12\,,\quad i=1,\dots n,$$

then $(-1)^{n-m}G(t,s)$ is a totally positive kernel.

Example 1. *The focal boundary value problem*

$$\begin{cases} x^{(n)}(t)+(-1)^{n-m}p(t)x(t)=f(t),\\ x^{(i)}(a)=0, & i=0,\dots,m-1,\\ x^{(i)}(b)=0, & i=m,\dots,n-1, \end{cases}$$

is uniquely solvable if $p(t)\ge0$, $t\in[a,b]$, $p\in\mathbf{L}[a,b]$.

Example 2. *Let $p^+\in\mathbf{L}[a,b]$, $p^+\ge0$, $t\in[a,b]$, and*

$$0\le p^-(t)\le\frac{24\cdot256}{27(b-a)^4}\,,\quad p^-(t)\not\equiv\frac{24\cdot256}{27(b-a)^4}\,. \tag{7}$$

Then the problem

$$\begin{cases} x^{(4)}(t)+p^+(t)x(t)-p^-(t)x(t)=f(t),\\ x(a)=0,\ \ \dot x(a)=0,\\ x(b)=0,\ \ \dot x(b)=0, \end{cases}$$

is uniquely solvable.

The constant in conditions (7) *are better than the constant $\frac{\pi^4}{(b-a)^4}$, which follows from Wirtinger's inequality.*

The conclusion: the classical results on totally positive kernels could be very useful for boundary value problems for ordinary differential equations.

Acknowledgements

The authors thank S. M. Labovskii for his help.

The work of E. Bravyi was performed as part of the State Task of the Ministry of Education and Science of the Russian Federation (project # 1.5336.2017/8.9) and was partially supported by the Russian Foundation for Basic Research (project # 18-01-00332).

References

[1] F. R. Gantmakher and M. G. Kreĭn, O*scillation Matrices and Kernels and Small Vibrations of Mechanical Systems.* (Russian) 2nd ed., Moscow, 1950.

[2] S. Karlin, *Total Positivity.* Vol. I. Stanford University Press, Stanford, Calif, 1968.

[3] S. Karlin, Total positivity, interpolation by splines, and Green's functions of differential operators. *J. Approximation Theory* **4** (1971), 91–112.

[4] A. Ju. Levin, *Selected Works.* (Russian) Yaroslavl'–Rybinsk, 2010.

[5] A. Ju. Levin and G. D. Stepanov, One-dimensional boundary value problems with operators that do not lower the number of sign changes. II. (Russian) *Sibirsk. Mat. Zh.* **17** (1976), no. 4, 813–830.