Boundary-Value Problems for Weakly Singular Integral Equations

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In the space $L_2[a,b]$, we consider a perturbed linear boundary-value problem for a weakly singular integral equation

$$x(t)-\int_a^b K(t,s)x(s)\,ds=f(t)+\varepsilon\int_a^b \overline{K}(t,s)x(s)\,ds, \tag{1}$$

$$lx(\,\cdot\,)=\alpha+\varepsilon Jx(\,\cdot\,). \tag{2}$$

We establish conditions for the bifurcation of solutions of the boundary-value problem (1), (2) and determine the structure of these solutions under the condition that the generating boundary-value problem

$$x(t)-\int_a^b K(t,s)x(s)\,ds=f(t),\ \ lx(\,\cdot\,)=\alpha \tag{3}$$

is unsolvable.

Here, $K(t,s)=\frac{H(t,s)}{|t-s|^\gamma}$ and $\overline{K}(t,s)=\frac{\overline{H}(t,s)}{|t-s|^\beta}$, where $H(t,s)$, $\overline{H}(t,s)$ are functions bounded in the domain $[a,b]\times[a,b]$, $0<\gamma<1$, $0<\beta<1$, $f\in L_2[a,b]$, $l=col(l_1,l_2,\dots,l_p):L_2[a,b]\to\mathbb{R}^p$, $J=col(J_1,J_2,\dots,J_p):L_2[a,b]\to\mathbb{R}^p$ are bounded linear functionals, $l_\nu,J_\nu:L_2[a,b]\to\mathbb{R}$, $\nu=\overline{1,p}$, $\alpha=col(\alpha_1,\alpha_2,\dots,\alpha_p)\in\mathbb{R}^p$ and $\varepsilon\ll 1$ is a small parameter.

By using the results obtained in [2], we show that the study of the problem of appearance of solutions of the boundary-value problem (1), (2) reduces to the corresponding task for the perturbed boundary-value problem for the Fredholm integral equation

$$x(t)=f_n(t)+\sum_{k=0}^{n}\varepsilon^k\int_a^b R_n^k(t,s)x(s)\,ds, \tag{4}$$

$$f_n(t)=f(t)+\sum_{k=1}^{n-1}\int_a^b R_k^0(t,s)f(s)\,ds+\sum_{k=1}^{n-1}\varepsilon^k\sum_{m=k}^{n-1}\int_a^b R_m^k(t,s)f(s)\,ds,$$

where $R_n^k(t,s)$, $k=\overline{0,n}$, are the sums of C_n^k kernels of all possible products of $n-k$ integral operators K and k integral operators $\overline{K}$

$$(Kw)(t)=\int_a^b\frac{H(t,s)}{|t-s|^\gamma}\,w(s)\,ds\ \text{ and }\ (\overline{K}w)(t)=\int_a^b\frac{\overline{H}(t,s)}{|t-s|^\beta}\,w(s)\,ds.$$

We apply the approach described in [3] to the study of the boundary-value problem (4), (2) and show that it can be reduced to the operator equation. Let $\{\varphi_i(t)\}_{i=1}^\infty$ be a complete orthonormal system of functions in $L_2[a,b]$. We introduce the notation

$$x_i=\int\limits_a^b x(t)\varphi_i(t)\,dt,\quad a_{ij}=\int\limits_a^b\int\limits_a^b K_n(t,s)\varphi_i(t)\varphi_j(s)\,dt\,ds,$$
$$a_{ij}^k=\int\limits_a^b\int\limits_a^b R_n^k(t,s)\varphi_i(t)\varphi_j(s)\,dt\,ds,\ \ k=\overline{1,n},$$
$$f_i=\int\limits_a^b f(t)\varphi_i(t)\,dt+\sum_{k=1}^{n-1}\int\limits_a^b\int\limits_a^b R_k^0(t,s)f(s)\varphi_i(t)\,dt\,ds,$$
$$f_i^k=\sum_{m=k}^{n-1}\int\limits_a^b\int\limits_a^b R_m^k(t,s)f(s)\varphi_i(t)\,ds\,dt,\quad k=\overline{1,n-1}.$$

By using this notation in the boundary-value problem (4), (2), we obtain the operator equation:

$$Uz=q+\sum_{k=1}^{n-1}\varepsilon^k q_k+\sum_{k=1}^{n}\varepsilon^k U_k z, \tag{5}$$

where

$$U=\begin{bmatrix}\Lambda\\ W\end{bmatrix},\quad U_1=\begin{bmatrix}\Lambda_1\\ W_1\end{bmatrix},\quad U_k=\begin{bmatrix}\Lambda_k\\ 0\end{bmatrix},\ \ k=\overline{2,n},$$
$$q=\begin{bmatrix}g\\ \alpha\end{bmatrix},\quad q_k=\begin{bmatrix}g_k\\ 0\end{bmatrix},\ \ k=\overline{1,n-1},$$

where the vectors z, g, g_k, $k=\overline{1,n-1}$ and the matrices W, W_1, Λ, Λ_k, $k=\overline{1,n}$ have the form

$$z=col\big(x_1,\ \ x_2,\ \ \dots,\ \ x_i,\ \ \dots\big),\quad g=col\big(f_1,\ \ f_2,\ \ \dots,\ \ f_i,\ \ \dots\big),$$
$$g_k=col\big(f_1^k,\ \ f_2^k,\ \ \dots,\ \ f_i^k,\ \ \dots\big),\quad W=l\Phi(\,\cdot\,),\quad W_1=J\Phi(\,\cdot\,),$$
$$\Lambda=\begin{pmatrix}1-a_{11} & -a_{12} & \dots & -a_{1i} & \dots\\ -a_{21} & 1-a_{22} & \dots & -a_{2i} & \dots\\ \dots & \dots & \dots & \dots & \dots\\ -a_{i1} & -a_{i2} & \dots & 1-a_{ii} & \dots\\ \dots & \dots & \dots & \dots & \dots\end{pmatrix},\quad \Lambda_k=\begin{pmatrix}a_{11}^k & a_{12}^k & \dots & a_{1i}^k & \dots\\ a_{21}^k & a_{22}^k & \dots & a_{2i}^k & \dots\\ \dots & \dots & \dots & \dots & \dots\\ a_{i1}^k & a_{i2}^k & \dots & a_{ii}^k & \dots\\ \dots & \dots & \dots & \dots & \dots\end{pmatrix},$$
$$\Phi(t)=\big(\varphi_1(t),\varphi_2(t),\dots,\varphi_i(t),\dots\big).$$

The generating equation for the operator equation (5) has the form

$$Uz=q. \tag{6}$$

The operator $\Lambda:\ell_2\to\ell_2$ appearing on the left-hand side of the operator equation (6) has the form $\Lambda=I-A$, where $I:\ell_2\to\ell_2$ is the identity operator and $A:\ell_2\to\ell_2$ is a compact operator. Hence, according to S. Krein's classification, the operator $\Lambda:\ell_2\to\ell_2$ is a Fredholm operator of index zero ($\dim\ker\Lambda=\dim\ker\Lambda^*<\infty$) and the operator $U:\ell_2\to\ell_2\times\mathbb{R}^p$ is a Fredholm operator of nonzero index ($\dim\ker U<\infty$, $\dim\ker U^*<\infty$).

The following statement is true for equation (6) (see [4]).

Theorem 1. *The homogeneous equation* (6) ($q=0$) *possesses a* d_2*-parameter family of solutions* $z\in\ell_2$,
$$z=P_{\Lambda_r}P_{Q_{d_2}}c_{d_2}\ \ \forall\,c_{d_2}\in\mathbb{R}^{d_2}.$$

The inhomogeneous equation (6) *is solvable if and only if the following* $r+d_1$ *linearly independent conditions are satisfied:*
$$P_{\Lambda_r^*}g=0,\quad P_{Q_{d_1}^*}(\alpha-W\Lambda^+g)=0,$$
and the equation possesses a d_2*-parameter family of solutions* $z\in\ell_2$ *of the form*
$$z=P_{\Lambda_r}P_{Q_{d_2}}c_{d_2}+P_{\Lambda_r}Q^+(\alpha-W\Lambda^+g)+\Lambda^+g\quad\forall c_{d_2}\in\mathbb{R}^{d_2}.$$

Here, $Q=WP_{\Lambda_r}$ is a $(p\times r)$-matrix, $P_{\Lambda_r}(P_{\Lambda_r^*})$ is a matrix formed by a complete system of r linearly independent columns (rows) of the matrix projector P_Λ (P_{Λ^*}), where P_Λ (P_{Λ^*}) is the projector onto the kernel (cokernel) of the matrix Λ, and $P_{Q_{d_2}}$ ($P_{Q_{d_1}^*}$) is a matrix formed by the complete system of d_2 (d_1) linearly independent columns (rows) of the matrix projector P_Q (P_{Q^*}), where P_Q (P_{Q^*}) is the projector onto the kernel (cokernel) of the matrix Q and Λ^+ (Q^+) is the pseudoinverse Moore–Penrose matrix for the matrix Λ (Q).

We now determine the conditions required for the bifurcation of solutions of the perturbed inhomogeneous boundary-value problem (1), (2) and study the structure of these solutions under the conditions that the solution of the homogeneous generating the boundary-value problem (3) ($f(t)=0$, $\alpha=0$) is not unique, i.e. (see [2]), $P_{\Lambda_r}P_{Q_{d_2}}\neq0$, and that the inhomogeneous generating boundary-value problem (3) is unsolvable.

It is known (see [9]) that small perturbations preserve the Fredholm property of the operator, i.e., the operator $\big(U-\sum\limits_{k=1}^{n}\varepsilon^kU_k\big)$ is a Fredholm operator with nonzero index. This enables one to investigate equation (5) by the methods of the theory of perturbed operator boundary-value problems with Fredholm linear part (see, e.g., [1,4,11]) obtained as a generalization of the classical methods of the perturbation theory of periodic boundary-value problems in the theory of oscillations (see [5,7,8,10]).

The analysis of the appearance of solutions of equation (5) is closely connected with the $((r+d_1)\times d_2)$-matrix
$$B_0=\begin{bmatrix}P_{\Lambda_r^*}\Lambda_1P_{\Lambda_r}P_{Q_{d_2}}\\ P_{Q_{d_1}^*}(W_1-W\Lambda^+\Lambda_1)P_{\Lambda_r}P_{Q_{d_2}}\end{bmatrix},$$
constructed by using the coefficients of equation (5).

We introduce an $((r+d_1)\times(r+d_1))$-matrix $P_{B_0^*}$, which is a projector onto the cokernel of the matrix B_0 and a matrix
$$G=\begin{bmatrix}-P_{\Lambda_r^*} & 0\\ P_{Q_{d_1}^*}W\Lambda^+ & -P_{Q_{d_1}^*}\end{bmatrix},$$
formed by $r+d_1$ rows and infinitely many columns. Moreover, as the matrix B_0, it is completely determined by the coefficients of equation (5).

By the Vishik–Lyusternik method (see [12]), we find efficient conditions for the coefficients guaranteeing the appearance of a family of solutions of the perturbed linear boundary-value problem (5) in the form of a Laurent series in powers of the small parameter ε with singularity at the point $\varepsilon=0$.

The results obtained for the perturbed equations (5) enable us to establish the conditions for the existence of a d_2-parameter family of solutions of the original perturbed boundary-value problem (1), (2). Indeed, if the boundary-value problem (1), (2) possesses at least one solution, then, according to the Riesz–Fischer theorem, one can find an element $x \in L_2[a,b]$ such that the quantities x_i, $i = \overline{1,\infty}$, determined from equation (5) are the Fourier coefficients of this elements, i.e., the following representation is true:

$$x(t) = \Phi(t)z. \tag{7}$$

As in [6], we conclude that the element $x(t)$ given by relations (7) is the required d_2-parameter family of solutions of the original boundary-value problem (1), (2). Therefore, the following statement is true.

Theorem 2. *Suppose that the generating boundary-value problem* (3) *is unsolvable. If conditions*

$$P_{\Lambda_r} P_{Q_{d_2}} \neq 0, \quad P_{B_0^*} G = 0,$$

are satisfied, then the boundary-value problem (1), (2) *has a d_2-parameter family of solutions in the form of series with singularity at the point $\varepsilon = 0$ convergent for sufficiently small fixed $\varepsilon \in (0, \varepsilon_*]$.*

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