Asymptotic Properties of Special Classes of Solutions of Second-Order Differential Equations with Nonlinearities Near to Regularly Varying

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The differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')\exp\big(R(|\ln|yy'||)\big), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\, \to]0, +\infty[$ $(-\infty < a < \omega \leq +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ are continuous functions, $Y_i \in \{0, \pm\infty\}$ $(i = 0, 1)$, Δ_{Y_i} is a one-sided neighborhood of Y_i, every function $\varphi_i(z)$ $(i = 0, 1)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order σ_i, $\sigma_0 + \sigma_1 \neq 1$, $\sigma_1 \neq 0$, the function $R :]0, +\infty[\, \to]0, +\infty[$ is continuously differentiable and regularly varying on infinity of the order μ, $0 < \mu < 1$, the derivative function of the function R is monotone, is considered in the work.

Definition. A solution y of equation (1) is called $P_\omega(Y_0, Y_1, \lambda_0)$ if it is defined on $[t_0, \omega[\, \subset [a, \omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

A lot of works (see, for example, [3, 4]) have been devoted to the establishing asymptotic representations of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equations of the form (1), in which $R \equiv 0$. The $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0 - 1}$ if $\lambda_0 \in R \setminus \{0, 1\}$. The asymptotic properties and necessary and sufficient conditions of existence of such solutions of equation (1) have been obtained in [1].

The case $\lambda_0 = \infty$ is one of the most difficult cases because in this case such solutions or their derivatives are slowly varying functions as $t \uparrow \omega$. Some results about asymptotic properties and existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) in the special case are presented in the work.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0; +\infty[$ satisfies the condition S if for any continuous differentiable function $L : \Delta_{Y_i} \to]0; +\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the next equality

$$\Theta(zL(z)) = \Theta(z)(1 + o(1)) \ \text{ is true as } \ z \to Y \ \ (z \in \Delta_Y)$$

holds.

Let us introduce the following notations

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t - \omega & \text{as } \ \omega < +\infty, \end{cases} \quad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i = 0, 1),$$

$$I(t) = \alpha_0 \int\limits_{A_\omega}^{t} p(\tau)\,d\tau, \quad A_\omega = \begin{cases} a & \text{if } \int\limits_a^\omega p(\tau)\,d\tau = +\infty, \\ \omega & \text{if } \int\limits_a^\omega p(\tau)\,d\tau < +\infty. \end{cases}$$

In case $\lim\limits_{t\uparrow\omega} |\pi_\omega(\tau)| \operatorname{sign} y_0^0 = Y_0$ we put

$$I_0(t) = \alpha_0 \int\limits_{A_\omega^0}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0} \Theta_0\big(|\pi_\omega(\tau)| \operatorname{sign} y_0^0\big)\,d\tau,$$

$$A_\omega^0 = \begin{cases} b_2 & \text{if } \int\limits_{b_2}^\omega p(t)|\pi_\omega(t)|^{\sigma_0} \Theta_0\big(|\pi_\omega(t)| \operatorname{sign} y_0^0\big)\,dt = +\infty, \\ \omega & \text{if } \int\limits_{b_2}^\omega p(t)|\pi_\omega(t)|^{\sigma_0} \Theta_0\big(|\pi_\omega(t)| y_0^0\big)\,dt < +\infty, \end{cases}$$

$$N(t) = \alpha_0 p(t)|\pi_\omega(t)|^{\sigma_0+1} \Theta_0\big(|\pi_\omega(t)| \operatorname{sign} y_0^0\big).$$

Here $b_1, b_2 \in [a;\omega[$ are chosen in such a way that $\frac{\operatorname{sign} y_0^1}{||\pi_\omega(t)|} \in \Delta_{Y_1}$ as $t \in [b_1;\omega]$ and $|\pi_\omega(\tau)| \operatorname{sign} y_0^0 \in \Delta_{Y_0}$ as $t \in [b_2;\omega]$.

The next three theorems are devoted to establishing $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1). First two cases are obtained in [2]. The first derivatives of such solutions are slowly varying functions as $t \uparrow \omega$, the fact creates difficulties in investigation of such solutions.

Theorem 1. *For the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation* (1) *the following conditions are necessary*

$$Y_0 = \begin{cases} \pm\infty & \textit{if } \omega = +\infty, \\ 0 & \textit{if } \omega < +\infty, \end{cases} \quad \pi_\omega(t) y_0^0 y_1^0 > 0 \ \textit{ as } \ t \in [a, \omega[\,. \tag{2}$$

If the function φ_0 satisfies the condition S and

$$\lim_{t\uparrow\omega} \frac{R'(|\ln|\pi_\omega(t)||) I_0(t)}{\pi_\omega(t) I_0'(t)} = 0, \tag{3}$$

then (2) *together with the next conditions are necessary and sufficient for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation* (1)*:*

$$\lim_{t\uparrow\omega} y_1^0 |I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}} = Y_1, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t) I_0'(t)}{I_0(t)} = 0, \quad y_1^0 (1-\sigma_0-\sigma_1) I_0(t) > 0 \ \textit{ as } \ t \in [b_2, \omega[\,.$$

For such solutions the next asymptotic representations take place as $t \uparrow \omega$

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t)) \exp(R(|\ln|y(t)||))} = (1-\sigma_0-\sigma_1) I_0(t)[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

Theorem 2. *If in* (1) *the function p is a continuously differentiable function, the function φ_0 satisfies the condition S and*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) N'(t)}{R'(|\ln|\pi_\omega(t)||) N(t)} = 0, \tag{4}$$

then together with (2) *the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions of equation* (1)*:*

$$\lim_{t\uparrow\omega} y_1^0 \exp\Big(\frac{1}{1-\sigma_0-\sigma_1}\,R\big(|\ln|\pi_\omega(t)||\big)\Big) = Y_1, \quad \alpha_0 y_1^0(1-\sigma_0-\sigma_1)\ln|\pi_\omega(t)| > 0 \ \text{ as } \ t\in[a,\omega[\,.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$

$$\frac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)y'(t)||))} = \frac{|1-\sigma_0-\sigma_1|N(t)}{R'(|\ln|\pi_\omega(t)||)}\,[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

Theorem 3. *If in* (1) *the function* p *is a continuously differentiable function, the function* φ_0 *satisfies the condition* S *and*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)} = M \neq 0, \tag{5}$$

then together with (2) *the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions of equation* (1)*:*

$$\lim_{t\uparrow\omega} y_1^0 \exp\Big(\frac{1}{1-\sigma_0-\sigma_1} R(|\ln|\pi_\omega(t)||)\Big) = Y_1, \quad \alpha_0 y_1^0(M+1)(1-\sigma_0-\sigma_1)\ln|\pi_\omega(t)| > 0 \ \text{ as } \ t\in[a,\omega[\,.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$

$$\frac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)y'(t)||))} = \frac{|1-\sigma_0-\sigma_1|N(t)(M+1)}{R'(|\ln|\pi_\omega(t)||)}\,[1+o(1)],$$
$$\frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

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