On Solvability of Poisson's Equation with Mixed Dirichlet and Nonlocal Integral Type Conditions

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Abstract. In this paper we study a problem for Poisson's equation when on opposite sides of rectangular domain are given the Dirichlet conditions, while on the rest two sides are given integral type nonlocal constraints. We prove the existence and uniqueness of a solution in the weighted Sobolev space.

Let $\Omega=\{x=(x_1,x_2):\ 0<x_k<l,\ k=1,2\}$ be a square with boundary Γ.
We seek in Ω a solution to Poisson's equation

$$\Delta u=-f(x),\ \ x\in\Omega, \tag{1}$$

which satisfies the following Dirichlet homogeneous conditions

$$u(x_1,0)=u(x_1,l)=0,\ \ 0\le x_1\le l, \tag{2}$$

and the integral type nonlocal conditions

$$\int\limits_0^\xi u(x)\,dx_1=0,\quad \int\limits_{l-\xi}^l u(x)\,dx_1=0,\ \ 0\le x_2\le l,\ \ 0<\xi\le\frac{l}{2}\,. \tag{3}$$

By $L_2(\Omega,\rho)$ we denote a weighted Lebesgue space of all real-valued functions $u(x)$ on Ω with the inner product and the norm

$$(u,v)_\rho=\int\limits_\Omega \rho uv\,dx,\ \ \|u\|_\rho=(u,u)_\rho^{1/2}.$$

Denote by $\overset{*}{W}{}_2^1(\Omega,\rho)$ a linear set of all functions $L_2(\Omega,\rho)$ whose first order derivatives (in general sense) belong to $L_2(\Omega,\rho)$. It is a normalized space with the norm

$$\|u\|_{1,\rho}=\big(\|u\|_\rho^2+|u|_{1,\rho}^2\big)^{1/2},\ \ |u|_{1,\rho}^2=\Big\|\frac{\partial u}{\partial x_1}\Big\|_\rho^2+\Big\|\frac{\partial u}{\partial x_2}\Big\|_\rho^2.$$

Let us choose a weight function $\rho(x)$ in the following form

$$\rho(x):=\begin{cases}\dfrac{x_1}{\xi}\,, & 0\le x_1\le\xi,\\ 1, & \xi<x_1<l-\xi,\\ \dfrac{l-x_1}{\xi}\,, & l-\xi\le x_1\le l,\end{cases}$$

and define an operator in the form

$$Gv(x):=\begin{cases}\dfrac{x_1}{\xi}\,v(x)-\dfrac{1}{\xi}\displaystyle\int\limits_0^{x_1}v(t,x_2)\,dt, & 0\le x_1\le\xi,\\ v(x), & \xi<x_1<l-\xi,\\ \dfrac{l-x_1}{\xi}\,v(x)-\dfrac{1}{\xi}\displaystyle\int\limits_{x_1}^{l}v(t,x_2)\,dt, & l-\xi\le x_1\le l.\end{cases}$$

We say that function $u\in\overset{*}{W}{}_2^1(\Omega,\rho)$ is a weak solution of problem (1)–(3) if the relation

$$a(u,v)=(f,Gv),\quad \forall\,v\in\overset{*}{W}{}_2^1(\Omega,\rho)\tag{4}$$

holds, where

$$a(u,v):=\Big(\frac{\partial u}{\partial x_1}\,,\frac{\partial v}{\partial x_1}\Big)_\rho+\Big(\frac{\partial u}{\partial x_2}\,,G\,\frac{\partial v}{\partial x_2}\Big).\tag{5}$$

Equation (4) can be formally obtained from (1) by taking into account conditions (2), (3).

In the case $u=v$ for estimate of the second addend of (5) we use the following proposition.

Lemma 1. *If the function v defined on the segment [0;1] satisfies the nonlocal conditions* (3)*, then the following identity*

$$\int\limits_0^l v(x)Gv(x)\,dx_1=\int\limits_0^l\rho(x)v^2(x)\,dx_1$$

holds.

Indeed, $v=\frac{\partial u}{\partial x_2}$ satisfies the nonlocal conditions. Besides, we take into account the equalities implied from the definition of the operator G

$$\int\limits_0^\xi v(x_1,x_2)\int\limits_0^{x_1}v(t,x_2)\;dt\,dx_1=\frac12\bigg(\int\limits_0^{x_1}v(t,x_2)\,dt\bigg)^2\bigg|_{x_1=0}^{\xi}=0,$$

$$\int\limits_{l-\xi}^{l} v(x_1,x_2)\int\limits_{x_1}^{l}v(t,x_2)\;dt\,dx_1=-\frac12\bigg(\int\limits_{x_1}^{l}v(t,x_2)\,dt\bigg)^2\bigg|_{x_1=l-\xi}^{l}=0.$$

In view of the following equalities

$$\frac{\partial}{\partial x_1}\,(Gv)=\rho\,\frac{\partial v}{\partial x_1}\,,\quad \int\limits_{\Omega}\frac{\partial^2 u}{\partial x_1^2}\,Gv\,dx_1\,dx_2=-\int\limits_{\Omega}\rho\,\frac{\partial u}{\partial x_1}\,\frac{\partial v}{\partial x_1}\,dx_1\,dx_2,$$

one can prove that the bilinear form $a(u,v)$ is both continuous and coercive on $u\in\overset{*}{W}{}_2^1(\Omega,\rho)$, while the linear form (f,Gv) is continuous on the same space.

Lemma 2. *For any function $u\in\overset{*}{W}{}_2^1(\Omega,\rho)$, the estimate*

$$\int\limits_0^l u^2\,dx_1\le\frac{5l^2}{4}\int\limits_0^l\rho\Big(\frac{\partial u}{\partial x_1}\Big)^2\,dx_1$$

is valid.

Proof. For simplicity let us write u' instead of $\partial u/\partial x_1$. Let

$$J:=-\int\limits_0^{\xi}x_1\,du^2+\int\limits_{\xi}^{l-\xi}\Big(\frac{l}{2}-x_1\Big)\,du^2+\int\limits_{l-\xi}^{l}(l-x_1)\,du^2. \tag{6}$$

It is easy to verify that

$$\int\limits_0^l u^2\,dx_1=J+\frac{l}{2}\,\big[u^2(\xi)+u^2(l-\xi)\big]. \tag{7}$$

Rewrite (6) as follows

$$J=-2\int\limits_0^{\xi}x_1u'u\,dx_1+2\int\limits_{\xi}^{l-\xi}\Big(\frac{l}{2}-x_1\Big)\,u'u\,dx_1+2\int\limits_{l-\xi}^{l}(l-x_1)u'u\,dx_1.$$

Whence, by use of ε-inequality, we obtain

$$\begin{aligned}|J|\le{}&\bigg[\frac{1}{2}\int\limits_0^{\xi}u^2\,dx_1+2\int\limits_0^{\xi}x_1^2(u')^2\,dx_1\bigg]+\bigg[\frac{1}{2}\int\limits_{\xi}^{l-\xi}u^2\,dx_1+2\int\limits_{\xi}^{l-\xi}\Big(\frac{l}{2}-x_1\Big)^2(u')^2\,dx_1\bigg]\\&+\bigg[\frac{1}{2}\int\limits_{l-\xi}^{l}u^2\,dx_1+2\int\limits_{l-\xi}^{l}(l-x_1)^2(u')^2\,dx_1\bigg].\end{aligned}$$

Now let us estimate the values $u(\xi)$, $u(l-\xi)$,

$$|\xi u(l-\xi)|^2=\bigg(\int\limits_{l-\xi}^{l}(l-x_1)u'\,dx_1\bigg)^2\le\frac{\xi^2}{2}\int\limits_{l-\xi}^{l}(l-x_1)(u')^2\,dx_1,$$

$$u^2(l-\xi)\le\frac{1}{2}\int\limits_{l-\xi}^{l}(l-x_1)(u')^2\,dx_1;\quad u^2(\xi)\le\frac{1}{2}\int\limits_0^{\xi}x_1(u')^2\,dx_1.$$

Finally, from (7) it follows

$$\frac{1}{2}\int\limits_0^l u^2\,dx_1 \le 2\int\limits_0^\xi x_1^2(u')^2\,dx_1 + 2\int\limits_\xi^{l-\xi}\Big(\frac{l}{2}-x_1\Big)^2(u')^2\,dx_1 + 2\int\limits_{l-\xi}^l (l-x_1)^2(u')^2\,dx_1 \\ + \frac{l}{4}\int\limits_0^\xi x_1(u')^2\,dx_1 + \frac{l}{4}\int\limits_{l-\xi}^l (l-x_1)(u')^2\,dx_1,$$

which confirms Lemma 2.

Thus, all the conditions of the Lax–Milgram lemma are fulfilled. Therefore, problem (1)–(3) has a unique weak solution from $\overset{*}{W}{}_2^1(\Omega,\rho)$.

References

[1] G. Berikelashvili, To a nonlocal generalization of the Dirichlet problem. *J. Inequal. Appl.* **2006**, Art. ID 93858, 6 pp.

[2] G. K. Berikelashvili and D. G. Gordeziani, On a nonlocal generalization of the biharmonic Dirichlet problem. (Russian) *Differ. Uravn.* **46** (2010), no. 3, 318–325; translation in *Differ. Equ.* **46** (2010), no. 3, 321–328.

[3] G. Berikelashvili and N. Khomeriki, On the convergence rate of a difference solution of the Poisson equation with fully nonlocal constraints. *Nonlinear Anal. Model. Control* **19** (2014), no. 3, 367–381.