Description of the Linear Perron Effect Under Parametric Perturbations Exponentially Decaying at Infinity

E. A. Barabanov

Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus
E-mail: bar@im.bas-net.by

V. V. Bykov

Lomonosov Moscow State University, Moscow, Russia
E-mail: vvbykov@gmail.com

1 Introduction

For a given integer $n \geq 2$ let $\mathcal{M}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1.1}$$

with continuous bounded coefficients defined on $\mathbb{R}_+$. Let us denote by $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ the Lyapunov exponents [7, p. 561], [1, p. 38] of system (1.1), by $\Lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$ their spectrum, and by $\mathrm{es}(A)$ its exponential stability index (i.e. the dimension of the linear subspace of solutions to this system that have negative characteristic exponents). In what follows, we identify system (1.1) with its defining function $A(\,\cdot\,)$ and therefore write $A \in \mathcal{M}_n$.

In his seminal paper [10] O. Perron constructed an example of a system $A \in \mathcal{M}_2$ for which there exists an exponentially decaying perturbation $Q : \mathbb{R}_+ \to \mathbb{R}^{2\times 2}$ such that the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \ \ x \in \mathbb{R}^2, \ \ t \in \mathbb{R}_+,$$

has the Lyapunov exponents

$$\lambda_2(A + Q) > \lambda_2(A) \ \text{ and } \ \lambda_1(A + Q) = \lambda_1(A). \tag{1.2}$$

The largest Lyapunov exponent $\lambda_2(A)$ of the unperturbed system A in Perron's example is positive, and hence this system is unstable and so is the perturbed one. In fact, the same example can be slightly modified to demonstrate the phenomenon of loss of stability in a linear system under exponentially decaying perturbation of its coefficients. Let $\sigma \in (\lambda_2(A), \lambda_2(A + Q))$. Then for the modified system $\widetilde{A} \equiv A - \sigma I_2$, where I_2 is the (2×2) identity matrix, we have

$$\mathrm{es}(\widetilde{A}) = 2 \ \text{ and } \ \mathrm{es}(\widetilde{A} + Q) = 1.$$

Thus, the system $\widetilde{A}$ is exponentially stable, whereas the perturbed system $\widetilde{A}+Q$ is only conditionally exponentially stable.

O. Perron also constructed [9] an example of a system $A \in \mathcal{M}_2$ with negative Laypunov exponents and its quadratic perturbation $f(x)$ such that the perturbed system $\dot{x} = A(t)x + f(x)$ possesses the following property: the characteristic exponent of any nontrivial solution starting at the line $x_1 = 0$ is the same as for the unperturbed system, while the characteristic exponent of any solution starting outside the line $x_1 = 0$ is greater than a certain positive number.

These examples by Perron served as a starting point for numerous studies of the effect of various classes of linear and nonliner perturbations on the Lyapunov exponents of systems in $\mathcal{M}_n$. The results obtained in this direction constitute an essential part of the modern theory of Lyapunov exponents. The effect of change of Lyapunov exponents of a system in $\mathcal{M}_n$ under one or another "small" perturbation was called in the monograph [6, Ch. 4] the Perron effect. Starting with the paper [5], the term is being used only for situations when perturbations do not decrease the Lyapunov exponents of the original system (in what follows, we will adhere to this terminology). Unlike the papers [5,6], which study the Perron effect under higher-order perturbations, and along the lines of the paper [10] we investigate linear vanishing at infinity perturbations of the coefficient matrix of a system in $\mathcal{M}_n$ and call this effect "linear Perron".

It is worth noting that the perturbation matrix constructed in paper [10] is of the form $Q(t) = \mu Q_0(t)$, where μ is a real parameter; it is established there that for each $\mu \neq 0$ relations(1.2) hold. With this in mind, given a metric space M we consider families of linear systems of the form

$$\dot{x} = (A(t) + Q(t,\mu))x, \ \ x \in \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \tag{1.3}$$

where $A \in \mathcal{M}_n$ and $Q(\,\cdot\,,\,\cdot\,) : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is jointly continuous matrix-valued function. For each fixed value of the parameter $\mu \in M$ we get a linear differential system with continuous bounded coefficients whose Lyapunov exponents will be denoted by $\lambda_1(\mu; A+Q) \leqslant \cdots \leqslant \lambda_n(\mu; A+Q)$. Therefore, the Lyapunov exponents of family (1.3) are functions of the parameter $\mu \in M$. In particular, the spectrum of family(1.3) is defined to be the vector function $\Lambda(\,\cdot\,; A+Q) \equiv (\lambda_1(\,\cdot\,; A+Q), \dots, \lambda_n(\,\cdot\,; A+Q)) : M \to \mathbb{R}^n$.

2 Statement of the problem. Main result

We will denote by $\mathcal{E}_n(M)$ the class of jointly continuous matrix-valued functions $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ satisfying the estimate

$$\|Q(t,\mu)\| \leqslant C_Q \exp(-\sigma_Q t), \ \ (t,\mu) \in \mathbb{R}_+ \times M,$$

with C_Q and σ_Q being positive constants (different for each function Q).

For a system $A \in \mathcal{M}_n$ we will denote by $\mathcal{E}_n[A](M)$ the class of those $Q \in \mathcal{E}_n(M)$ that do not descrease its Lyapunov exponents, i.e. for any $A \in \mathcal{M}_n$ and its perturbation $Q \in \mathcal{E}_n[A](M)$ the inequalities

$$\inf_{\mu\in M} \lambda_i(\mu; A+Q) \geqslant \lambda_i(A), \ \ i = 1, \dots, n,$$

hold. Clearly, for any $A \in \mathcal{M}_n$ the class $\mathcal{E}_n[A](M)$ is nonempty since identically zero matrix belongs to it.

The problem to be solved is to obtain for each $n \geq 2$ and each metric space M a complete description of the class of pairs $(\Lambda(A), \Lambda(\,\cdot\,; A+Q))$ composed of the spectrum $\Lambda(A) \in \mathbb{R}^n$ of a system $A \in \mathcal{M}_n$ and the spectrum $\Lambda(\,\cdot\,; A+Q) : M \to \mathbb{R}^n$ of a family $A+Q$, where A ranges over $\mathcal{M}_n$ and matrix-valued function Q ranges over the class $\mathcal{E}_n[A](M)$ for each A, i.e. of the class

$$\Pi\mathcal{E}_n(M) = \Big\{(\Lambda(A), \Lambda(\,\cdot\,; A+Q)) \ | \ \ A \in \mathcal{M}_n, \ \ Q \in \mathcal{E}_n[A](M)\Big\}.$$

Note that a complete description of the class

$$\Lambda\mathcal{E}_n(M) = \Big\{\Lambda(\,\cdot\,; A+Q) \ | \ \ A \in \mathcal{M}_n, \ \ Q \in \mathcal{E}_n[A](M)\Big\},$$

composed of the second elements of the pairs in the class $\Pi\mathcal{E}_n(M)$ immediately follows from the result of [3].

Obviously, the solution of the stated problem would contain as a special case Perron's example and describe from a descriptive set theoretic standpoint all possible situations in which an exponentially stable linear system gets unstable under parametric exponentially vanishing perturbations. For instance, it follows from the theorem presented below that there exists a system $A \in \mathcal{M}_2$ with the largest Laypunov exponent $\lambda_2(A) = -1$ and its perturbation $Q \in \mathcal{E}_2[A](\mathbb{R})$ such that the largest Laypunov exponent $\lambda_2(A+Q)$ of the perturbed system equals -1 for a rational μ and 1 for an irrational μ.

The direction in the theory of Lyapunov exponents dealing with the dependence of asymptotic properties and characteristics of parametric differential systems on the parameter is due to V. M. Millionshchikov, who initiated systematic research in this direction with a series of papers, of which we only mention the paper [8]. We are also indebted to him for understanding that the language of the Baire theory of discontinuous functions is adequate for describing such a dependence. We emphasize that here one speaks of a complete description of all possible types of behavior of some properties or characteristics of a system under changes in the system parameters as opposed to establishing sufficient conditions for one or another type of their behavior. Since then, quite a few results have been obtained in this vein.

Let us recall that a function $f : M \to \mathbb{R}$ is said [4, pp. 266–267] to be of the class $({}^*, G_\delta)$ if for each $r \in \mathbb{R}$ the preimage $f^{-1}([r, +\infty))$ of the half-interval $[r, +\infty)$ is a G_δ-set of the metric space M. In particular, the class $({}^*, G_\delta)$ is a subclass of the second Baire class [4, p. 294].

A complete description of the class $\Pi\mathcal{E}_n(M)$ for any $n \geq 2$ and metric space M is given by the following statement [2].

Theorem. *Let $n \geqslant 2$ be an integer and M a metric space. A pair $(l, F(\,\cdot\,))$, with $l = (l_1, \dots, l_n) \in \mathbb{R}^n$ and $F(\,\cdot\,) = (f_1(\,\cdot\,), \dots, f_n(\,\cdot\,)) : M \to \mathbb{R}^n$, belongs to the class $\Pi\mathcal{E}_n(M)$ if and only if the following conditions are met:*

(1) $l_1 \leqslant \cdots \leqslant l_n$;

(2) $f_1(\mu) \leqslant \cdots \leqslant f_n(\mu)$ *for all* $\mu \in M$;

(3) $f_i(\mu) \geqslant l_i$ *for all* $\mu \in M$ *and* $i = 1, \dots, n$;

(4) *for each* $i = 1, \dots, n$ *the function* $f_i(\,\cdot\,) : M \to \mathbb{R}$ *is bounded and is of the class* $({}^*, G_\delta)$.

Corollary 2.1. *Let $n \geqslant 2$ be an integer and M an interval in the real line. Then for each pair $(l, F(\,\cdot\,)) \in \Pi\mathcal{E}_n(M)$ there exists a system $A \in \mathcal{M}_n$ and its perturbation $Q \in \mathcal{E}_n[A](\mathbb{R})$ analytical in parameter such that $\Lambda(A) = l$ and $\Lambda(\,\cdot\,; A+Q) = F$.*

Let $\mathcal{Z}_n \equiv \{0, \dots, n\}$. We define the function $\mathrm{es}(\,\cdot\,; A) : M \to \mathcal{Z}_n$ assigning to each $\mu \in M$ the exponential stability index of system (1.3). There naturally arises the problem of describing the class of pairs composed of the exponential stability index $\mathrm{es}(A) \in \mathcal{Z}_n$ of a system A and the exponential stability index $\mathrm{es}(\,\cdot\,; A+Q) : M \to \mathcal{Z}_n$ of a family $A+Q$, i.e. of the class

$$\mathcal{IE}_n(M) = \Big\{ (\mathrm{es}(A), \mathrm{es}(\,\cdot\,; A+Q)) \mid \ A \in \mathcal{M}_n, \ \ Q \in \mathcal{E}_n[A](M) \Big\}.$$

The solution is provided by the following statement.

Corollary 2.2. *Let $n \geqslant 2$ be an integer and M a metric space. A pair $(d, f(\,\cdot\,))$, where $d \in \mathcal{Z}_n$ and $f : M \to \mathcal{Z}_n$, belongs to the class $\mathcal{IE}_n(M)$ if and only if $f(\mu) \leqslant d$ for all $\mu \in M$ and the function $(-f)$ is of the class $({}^*, G_\delta)$.*

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