On Generic Inhomogeneous Boundary-Value Problems for Differential Systems in Sobolev spaces

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Let a finite interval $[a,b]\subset\mathbb{R}$ and parameters $\{m,n,r,l\}\subset\mathbb{N}$, $1\le p\le\infty$, be given. By $W_p^{n+r}=W_p^{n+r}([a,b];\mathbb{C}):=\{y\in C^{n+r-1}[a,b]:\ y^{(n+r-1)}\in AC[a,b],\ y^{(n+r)}\in L_p[a,b]\}$ we denote a complex Sobolev space and set $W_p^0:=L_p$. This space is a Banach one with respect to the norm

$$\|y\|_{n+r,p}=\sum_{k=0}^{n+r-1}\|y^{(k)}\|_p+\|y^{(n+r)}\|_p,$$

where $\|\cdot\|_p$ is the norm in space $L_p([a,b];\mathbb{C})$. Similarly, by $(W_p^{n+r})^m:=W_p^{n+r}([a,b];\mathbb{C}^m)$ and $(W_p^{n+r})^{m\times m}:=W_p^{n+r}([a,b];\mathbb{C}^{m\times m})$ we denote Sobolev spaces of vector-valued functions and matrix-valued functions, respectively, whose elements belong to the function space W_p^{n+r}.

We consider the following linear boundary-value problem

$$(Ly)(t):=y^{(r)}(t)+\sum_{j=1}^{r}A_{r-j}(t)y^{(r-j)}(t)=f(t),\ \ t\in(a,b), \tag{1}$$

$$By=c, \tag{2}$$

where the matrix-valued functions $A_{r-j}(\cdot)\in(W_p^n)^{m\times m}$, the vector-valued function $f(\cdot)\in(W_p^n)^m$, the vector $c\in\mathbb{C}^l$, the linear continuous operator

$$B:(W_p^{n+r})^m\to\mathbb{C}^l \tag{3}$$

are arbitrarily chosen; and the vector-valued function $y(\cdot)\in(W_p^{n+r})^m$ is unknown.

We represent vectors and vector-valued functions in the form of columns. A solution to the boundary-value problem (1), (2) is understood as a vector-valued function $y(\cdot)\in(W_p^{n+r})^m$ satisfying equation (1) almost everywhere on (a,b) (everywhere for $n\ge 2$) and equality (2) specifying l scalar boundary conditions. The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the boundary condition (2) with continuous operator (3) is the most general condition for this equation.

It includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and multi-point problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives (generally fractional) $y^{(k)}(\cdot)$ with $0<k\le n+r$.

For $1\le p<\infty$, every operator B in (3) admits a unique analytic representation

$$By=\sum_{k=0}^{n+r-1}\alpha_k y^{(k)}(a)+\int_a^b\Phi(t)y^{(n+r)}(t)\,dt,\ \ y(\cdot)\in(W_p^{n+r})^m,$$

where the matrices $\alpha_k\in\mathbb{C}^{rm\times m}$ and the matrix-valued function $\Phi(\cdot)\in L_{p'}([a,b];\mathbb{C}^{rm\times m})$, $1/p+1/p'=1$.

For $p = \infty$ this formula also defines an operator $B : (W_\infty^{n+r})^m \to \mathbb{C}^{rm}$. However, there exist other operators from this class generated by the integrals over finitely additive measures.

With the generic inhomogeneous boundary-value problem (1), (2), we associate a linear continuous operator in pair of Banach spaces

$$(L, B) : (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l. \tag{4}$$

Recall that a linear continuous operator $T\colon X \to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If operator T is Fredholm, then its range $T(X)$ is closed in Y and the index

$$\operatorname{ind} T := \dim \ker T - \dim(Y/T(X)) \in \mathbb{Z}$$

is finite.

Theorem 1. *The linear operator* (4) *is a bounded Fredholm operator with index $mr - l$.*

Theorem 1 allows the next specification.

For each number $k \in \{1, \dots, r\}$, we consider the family of the matrix Cauchy problems:

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \ \ t \in (a, b),$$

with the initial conditions

$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \ \ j \in \{1, \dots, r\}.$$

Here, $Y_k(\,\cdot\,)$ is an unknown $m \times m$ matrix-valued function, and $\delta_{k,j}$ is the Kronecker symbol.

By $[BY_k]$ we denote the numerical $m \times l$ matrix, in which j-th column is the result of action of the operator B on the j-th column of the matrix-valued function $Y_k(\,\cdot\,)$.

Definition 1. A block rectangular numerical matrix $M(L, B) := ([BY_0], \dots, [BY_{r-1}]) \in \mathbb{C}^{mr \times l}$ is characteristic to the inhomogeneous boundary-value problem (1), (2). It consists of r rectangular block columns $[BY_k(\,\cdot\,)] \in \mathbb{C}^{m \times l}$.

Here mr is the number of scalar differential equations of system (1), and l is the number of scalar boundary conditions.

Theorem 2. *The dimensions of the kernel and cokernel of operator* (4) *are equal to the dimensions of the kernel and cokernel of the characteristic matrix $M(L, B)$, respectively.*

Theorem 2 implies a criterion for the invertibility of the operator (4).

Corollary 1. *The operator (L, B) is invertible if and only if $l = mr$ and the matrix $M(L, B)$ is nondegenerate.*

With problem (1), (2), we consider the sequence of boundary-value problems

$$L(k)y(t, k) := y^{(r)}(t, k) + \sum_{j=1}^{r} A_{r-j}(t, k) y^{(r-j)}(t, k) = f(t, k), \ \ t \in (a, b), \tag{5}$$

$$B(k)y(\,\cdot\,, k) = c(k), \ \ k \in \mathbb{N}, \tag{6}$$

where the matrix-valued functions $A_{r-j}(\,\cdot\,, k)$, the vector-valued function $f(\,\cdot\,, k)$, the vector $c(k)$, and a linear continuous operator $B(k)$ satisfy the above conditions to problem (1), (2).

With the boundary-value problem (5), (6), we associate a sequence of linear continuous operators

$$(L(k), B(k)) : (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l$$

and a sequence of characteristic matrices depending on the parameter $k \in \mathbb{N}$

$$M(L(k), B(k)) := \Big(\big[B(k)Y_0(\,\cdot\,,(k))\big], \dots, \big[B(k)Y_{r-1}(\,\cdot\,,(k))\big] \Big) \subset \mathbb{C}^{mr\times l}.$$

We now formulate a sufficient condition for the convergence of the characteristic matrices $M(L(k), B(k))$ to the matrix $M(L, B)$.

Theorem 3. *If the sequence of operators $(L(k), B(k))$ converges strongly to the operator (L, B) for $k \to \infty$, then the sequence of characteristic matrices $M(L(k), B(k))$ converges to the matrix $M(L, B)$.*

Theorem 3 implies the next result.

Corollary 2. *Under the assumptions from Theorem 3, the following inequalities hold for sufficiently large k:*

$$\dim\ker(L(k), B(k)) \le \dim\ker(L, B),$$
$$\dim\operatorname{coker}(L(k), B(k)) \le \dim\operatorname{coker}(L, B).$$

In particular:

1. If $l = mr$ and the operator (L, B) is reversible, then the operators $(L(k), B(k))$ are also reversible for large k.
2. If the boundary-value problem (1), (2) has a solution for any values of the right-hand sides, then the boundary-value problems (5), (6) also have a solution for large k.
3. If the boundary-value problem (1), (2) has a unique solution, then the problems (5), (6) also have a unique solution for each sufficiently large k.

Let us consider parameterized by number $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, linear boundary-value problem

$$L(\varepsilon)y(t,\varepsilon) := y^{(r)}(t,\varepsilon) + \sum_{j=1}^{r} A_{r-j}(t,\varepsilon)y^{(r-j)}(t,\varepsilon) = f(t,\varepsilon), \ \ t \in (a,b), \tag{7}$$

$$B(\varepsilon)y(\,\cdot\,;\varepsilon) = c(\varepsilon), \tag{8}$$

where for every fixed ε the matrix-valued functions $A_{r-j}(\,\cdot\,;\varepsilon) \in (W_p^n)^{m\times m}$, the vector-valued function $f(\,\cdot\,;\varepsilon) \in (W_p^n)^m$, the vector $c(\varepsilon) \in \mathbb{C}^{rm}$, $B(\varepsilon)$ is the linear continuous operator $B(\varepsilon) : (W_p^{n+r})^m \to \mathbb{C}^{rm}$, and the solution (the unknown vector-valued function) $y(\,\cdot\,;\varepsilon) \in (W_p^{n+r})^m$.

It follows from Theorem 1 that the boundary-value problem (7), (8) is a Fredholm one with index zero.

Definition 2. A solution to the boundary-value problem (7), (8) depends continuously on the parameter ε at $\varepsilon = 0$ if the following two conditions are satisfied:

($*$) there exists a positive number $\varepsilon_1 < \varepsilon_0$ such that for any $\varepsilon \in [0, \varepsilon_1)$ and arbitrary chosen right-hand sides $f(\,\cdot\,;\varepsilon) \in (W_p^n)^m$ and $c(\varepsilon) \in \mathbb{C}^{rm}$ this problem has a unique solution $y(\,\cdot\,;\varepsilon)$ that belongs to the space $(W_p^{n+r})^m$;

$(**)$ the convergence of the right-hand sides $f(\,\cdot\,;\varepsilon) \to f(\,\cdot\,;0)$ in $(W_p^n)^m$ and $c(\varepsilon) \to c(0)$ in $\mathbb{C}^{rm}$ as $\varepsilon \to 0+$ implies the convergence of the solutions $y(\,\cdot\,;\varepsilon) \to y(\,\cdot\,;0)$ in $(W_p^{n+r})^m$.

Consider the following conditions as $\varepsilon \to 0+$:

(0) the limiting homogeneous boundary-value problem

$$L(0)y(t,0) = 0, \ \ t \in (a,b), \ \ B(0)y(\,\cdot\,,0) = 0$$

has only the trivial solution;

(I) $A_{r-j}(\,\cdot\,;\varepsilon) \to A_{r-j}(\,\cdot\,;0)$ in the space $(W_p^n)^{m\times m}$ for each number $j \in \{1,\dots,r\}$;

(II) $B(\varepsilon)y \to B(0)y$ in the space $\mathbb{C}^{rm}$ for every $y \in (W_p^{n+r})^m$.

Theorem 4. *A solution to the boundary-value problem* (7), (8) *depends continuously on the parameter ε at $\varepsilon = 0$ if and only if this problem satisfies condition* (0) *and the conditions* (I) *and* (II).

We supplement our result with a two-sided estimate of the error $\|y(\,\cdot\,;0) - y(\,\cdot\,;\varepsilon)\|_{n+r,p}$ of the solution $y(\,\cdot\,;\varepsilon)$ via its discrepancy

$$\widetilde{d}_{n,p}(\varepsilon) := \big\|L(\varepsilon)y(\,\cdot\,;0) - f(\,\cdot\,;\varepsilon)\big\|_{n,p} + \big\|B(\varepsilon)y(\,\cdot\,;0) - c(\varepsilon)\big\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\,\cdot\,;0)$ as an approximate solution to problem (7), (8).

Theorem 5. *Suppose that the boundary-value problem* (7), (8) *satisfies conditions* (0), (I) *and* (II). *Then there exist positive numbers $\varepsilon_2 < \varepsilon_1$ and γ_1, γ_2 such that, for any $\varepsilon \in (0,\varepsilon_2)$, the following two-sided estimate is true:*

$$\gamma_1\,\widetilde{d}_{n,p}(\varepsilon) \le \big\|y(\,\cdot\,;0) - y(\,\cdot\,;\varepsilon)\big\|_{n+r,p} \le \gamma_2\,\widetilde{d}_{n,p}(\varepsilon),$$

where the quantities ε_2, γ_1, and γ_2 do not depend of $y(\,\cdot\,;\varepsilon)$ and $y(\,\cdot\,;0)$.

Thus, the error and discrepancy of the solution $y(\,\cdot\,;\varepsilon)$ to the boundary-value problem (7), (8) are of the same degree of smallness.

The results are published in [1, 3–7]. The most general class of multi-point boundary-value problems for systems of linear ordinary differential equations of an arbitrary order is considered in [2].

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