Existence and Uniqueness Theorems to Generalized Emden–Fowler Type Equations

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Abstract. For generalized Emden–Fowler type equations we obtain conditions on initial values providing uniqueness or non-uniqueness of solutions.

1 Introduction and Basic Notation

Consider the equation

$$y''=p(x,y,y')|y|_{\pm}^{k_0}|y'|_{\pm}^{k_1}, \tag{1.1}$$

where $|a|_{\pm}^{b}$ denotes $|a|^b \operatorname{sgn} a$ and a positive continuous function p is locally Lipschitz continuous in the last two arguments. The real constants k_0 and k_1 are positive.

Given any $x_0, y_0, y_1 \in \mathbf{R}$, equation (1.1) has a solution defined in a neighborhood of $x_0 \in \mathbf{R}$ and satisfying the initial conditions

$$y(x_0)=y_0, \quad y'(x_0)=y_1. \tag{1.2}$$

Our purpose is to know whether or not the above solution is unique. To obtain results, we use some methods of [1]. In some simple cases the results coincide with those of [2] and [3].

Without loss of generality, suppose $x_0=0$. Put

$$\begin{gathered}p_0=p(0,0,0)>0,\\ p_m(X)=\inf\big\{p(x,u,v):\ |x|\le X,\ |u|\le X,\ |v|\le X\big\},\\ p_M(X)=\sup\big\{p(x,u,v):\ |x|\le X,\ |u|\le X,\ |v|\le X\big\},\end{gathered}$$

and note that $p_m(X)\to p_0$ and $p_M(X)\to p_0$ as $X\to +0$.

Since p is locally Lipschitz continuous in the last two arguments, we may assume it to satisfy the inequalities

$$|p(x,u,v)-p(x,w,v)|\le p_0\lambda_X|u-w| \ \text{ and } \ |p(x,u,v)-p(x,u,w)|\le p_0\lambda_X|v-w|$$

for some $\lambda_X>0$ and for all real $x,u,v,w\in[-X;X]$.

2 Main Results

Theorem 2.1. *If* $k_0\in(0;1)$, $y_0=0$, $y_1\neq 0$, *then in a neighborhood of* 0 *equation* (1.1) *has a unique solution satisfying* (1.2).

Theorem 2.2. *If $k_1 \in (0;1)$, $y_0 \neq 0$, $y_1 = 0$, then equation* (1.1) *has at least two solutions satisfying* (1.2) *and differing at points arbitrarily close to* 0.

Theorem 2.3. *If $k_0 > 0$, $k_1 > 0$, $k_0 + k_1 \geq 1$, $y_0 = y_1 = 0$, then in a neighborhood of* 0 *equation* (1.1) *has a unique solution satisfying* (1.2).

Theorem 2.4. *If $k_0, k_1, k_0 + k_1 \in (0;1)$ and $y_0 = y_1 = 0$, then in a neighborhood of* 0 *equation* (1.1) *has at least two solutions satisfying* (1.2) *and differing at points arbitrarily close to* 0.

3 Proofs

Proof of Theorem 2.1. According to the equation and initial conditions, we have

$$y(x) = \int\limits_0^x y'(\xi)\,d\xi \ \text{ and } \ y'(x) = y_1 + \int\limits_0^x p\Big(\eta, \int\limits_0^\eta y'(\xi)\,d\xi,\, y'(\eta)\Big)\Big|\int\limits_0^\eta y'(\xi)\,d\xi\Big|_\pm^{k_0} |y'(\eta)|_\pm^{k_1}\,d\eta.$$

The last expression can be written as $F(y',y',y',y')(x)$, where

$$F(u_1,u_2,u_3,u_4)(x) = y_1 + \int\limits_0^x p\Big(\eta, \int\limits_0^\eta u_1(\xi)\,d\xi, u_2(\eta)\Big)\Big|\int\limits_0^\eta u_3(\xi)\,d\xi\Big|_\pm^{k_0} |u_4(\eta)|_\pm^{k_1}\,d\eta$$

for any continuous functions u_1, u_2, u_3, u_4.

Suppose y and z are different solutions to (1.1), (1.2). There exists a segment $[-X;X]$ with $0 < X < 1$ such that both $y'(x)/y_1$ and $z'(x)/y_1$ are contained in $[\frac12\,;2]$ for any $x \in [-X;X]$.

Put $\delta = \sup\{|y'(x) - z'(x)| :\ x \in [-X;X]\}$. We have

$$\begin{aligned}
|y'(x) - z'(x)| &= \big|F(y',y',y',y')(x) - F(z',z',z',z')(x)\big| \\
&\leq \big|F(y',y',y',y')(x) - F(y',y',y',z')(x)\big| + \big|F(y',y',y',z')(x) - F(y',y',z',z')(x)\big| \\
&\quad + \big|F(y',y',z',z')(x) - F(y',z',z',z')(x)\big| + \big|F(y',z',z',z')(x) - F(z',z',z',z')(x)\big|.
\end{aligned}$$

Now we estimate, on $[-X;X]$, each summand of the last sum. For the second one, we use the inequality

$$\big||a|_\pm^k - |b|_\pm^k\big| \leq \frac{k|a-b|}{\min\{|a|,|b|\}^{1-k}} \ \text{ whenever } \ 0 < k < 1 \ \text{ and } \ \operatorname{sgn} a = \operatorname{sgn} b \neq 0.$$

So,

$$\begin{aligned}
\big|F(y',y',y',y')(x) - F(y',y',y',z')(x)\big| &\leq X \cdot p_M(X) \cdot |2y_1X|^{k_0} \cdot k_1|y_1|^{k_1-1}2^{|k_1-1|}\delta, \\
\big|F(y',y',y',z')(x) - F(y',y',z',z')(x)\big| &\leq p_M(X) \cdot \frac{k_0}{k_0+1}\, X^{k_0+1}\Big|\frac{2}{y_1}\Big|^{1-k_0}\delta \cdot |2y_1|^{k_1}, \\
\big|F(y',y',z',z')(x) - F(y',z',z',z')(x)\big| &\leq X \cdot p_0\lambda_X\delta \cdot |2y_1X|^{k_0} \cdot |2y_1|^{k_1}, \\
\big|F(y',z',z',z')(x) - F(z',z',z',z')(x)\big| &\leq X \cdot Xp_0\lambda_X\delta \cdot |2y_1X|^{k_0} \cdot |2y_1|^{k_1}.
\end{aligned}$$

Now we choose $X > 0$ small enough to make each right-hand side of the four inequalities less than $\delta/8$. This yields $|y'(x) - z'(x)| < \delta/2$ on $[-X;X]$, contradicting to the definition of δ. □

Proof of Theorem 2.2. Without loss of generality we assume $y_0 > 0$.

The first solution to (1.1), (1.2) is evident: $y \equiv y_0$. To find another one, put $\alpha = \frac{1}{1-k_1} > 1$ and consider the first-order 2-dimensional system

$$\begin{cases} y'(x) = |v(x)|^{\alpha}, \\ v'(x) = \dfrac{|y(x)|_{\pm}^{k_0}}{\alpha}\, p\big(x, y(x), |v(x)|^{\alpha}\big) \end{cases}$$

with the initial conditions $y(0) = y_0$, $v(0) = 0$.

Since $y_0 \neq 0$, this initial value problem is regular in a neighborhood of the point $(0, y_0, 0)$ regardless of whether or not k_0 is less than 1. Hence the problem has a solution defined in a neighborhood of 0. It follows from the second equation of the system that $v'(0) \neq 0$ and therefore $y'(x)$, which equals $|v(x)|^{\alpha}$, vanishes at 0 but cannot be identically zero in any neighborhood of 0. So, y cannot be constant.

Further, $y(x)$, $v(x)$, and $y'(x)$ are positive for $x > 0$ and

$$y''(x) = \alpha v(x)^{\alpha-1}\, \frac{y(x)^{k_0}}{\alpha}\, p(x, y(x), v(x)^{\alpha}) = y'(x)^{(\alpha-1)/\alpha}\, y(x)^{k_0} p(x, y(x), y'(x)).$$

Since $(\alpha-1)/\alpha = k_1$, the function $y(x)$ is a solution to (1.1), (1.2) other than the constant one. □

Proof of Theorem 2.3. The existence of a solution is evident even without the Peano existence theorem since $y \equiv 0$ surely satisfies both (1.1) and (1.2). So, we have to prove that no other solution exists in a sufficiently small neighborhood of 0.

First, consider constant-sign solutions to (1.1), (1.2) with constant-sign derivative in a half-neighborhood of 0. Here we have the following equivalences for such solutions (as $x \to 0$):

$$y''(x)|y'(x)|^{1-k_1} \sim p_0\, |y(x)|_{\pm}^{k_0}\, y'(x),$$

$$\begin{cases} \big(\log|y'|\operatorname{sgn} y'\big)'(x) \sim \dfrac{p_0}{k_0+1}\, \big(|y|^{k_0+1}\big)'(x) & \text{if } k_1 = 2, \\ \big(|y'|_{\pm}^{2-k_1}\big)'(x) \sim \dfrac{(2-k_1)p_0}{k_0+1}\, \big(|y|^{k_0+1}\big)'(x) & \text{if } k_1 \neq 2. \end{cases}$$

The right-hand sides of the two last equivalences are the derivatives of bounded functions. The same must be true for equivalent functions. But in the case $k_1 \geq 2$, the left-hand sides are the derivatives of unbounded functions. Because of this contradiction, we go on with the case $k_1 < 2$ only. By L'Hôpital's rule, the last equivalence invokes

$$|y'(x)|_{\pm}^{2-k_1} \sim \frac{(2-k_1)p_0}{k_0+1}\, |y(x)|^{k_0+1},$$

$$y'(x) \sim \Big(\frac{(2-k_1)p_0}{k_0+1}\Big)^{1/(2-k_1)} |y(x)|^{(k_0+1)/(2-k_1)},$$

$$\begin{cases} \big(\log|y|\, \operatorname{sgn} y\big)'(x) \sim \Big(\dfrac{(2-k_1)p_0}{k_0+1}\Big)^{1/(2-k_1)} & \text{if } k_0+1 = 2-k_1, \\ \big(|y|_{\pm}^{1-(k_0+1)/(2-k_1)}\big)'(x) \sim \Big(\dfrac{(2-k_1)p_0}{k_0+1}\Big)^{1/(2-k_1)} \Big(1 - \dfrac{k_0+1}{2-k_1}\Big) & \text{if } k_0+1 \neq 2-k_1. \end{cases}$$

By the conditions of the theorem, the exponent of $|y|_{\pm}$ in the last equivalence, which equals

$$1 - \frac{k_0+1}{2-k_1} = \frac{1-k_0-k_1}{2-k_1}\,,$$

is negative. Hence, the left-hand sides of the last two equivalences are the derivatives of unbounded functions but are equivalent to finite constants. This contradiction shows that in any half-neighborhood of 0 there is no constant-sign solution to (1.1), (1.2) with constant-sign derivative, besides the trivial solution $y \equiv 0$.

Now, what about non-constant-sign solutions? If such a solution pretends to disprove the statement of the theorem, its domain must include a monotonic sequence of disjoint intervals $(a_j; b_j)$ such that

(i) $y(x)y'(x) \neq 0$ on $(a_j; b_j)$,

(ii) $y(a_j)y'(a_j) = 0$,

(iii) $y(b_j)y'(b_j) = 0$,

(iv) $a_j \to 0$ and $b_j \to 0$ as $j \to \infty$.

Note that neither $y(a_j) = y'(a_j) = 0$ nor $y(b_j) = y'(b_j) = 0$ can hold because of the first part of our proof. Neither $y(a_j) = y(b_j) = 0$ nor $y'(a_j) = y'(b_j) = 0$ can hold because of condition (i), Rolle's lemma, and equation (1.1). If $y(a_j) = 0$ and $y'(a_j) > 0$, then, according to (1.1), we have $y(x) > 0$, $y'(x) > 0$, and $y''(x) > 0$ on $(a_j; b_j)$, which makes (iii) impossible. Similarly, if $y(b_j) = 0$ and $y'(b_j) > 0$, then we have $y(x) < 0$, $y'(x) > 0$, and $y''(x) < 0$ on $(a_j; b_j)$, which also makes (iii) impossible. So, only the cases $y(a_j) = 0$, $y'(a_j) < 0$, $y(b_j) < 0$, $y'(b_j) = 0$ and $y(a_j) > 0$, $y'(a_j) = 0$, $y(b_j) = 0$, $y'(b_j) < 0$ are possible. A pair of such segments can match at a common end-point with $y(x) = 0$. But outside their union the solution can only stay constant or move away from zero. Thus, it cannot satisfy (1.2). □

Proof of Theorem 2.4. The first solution to (1.1), (1.2) is $y \equiv 0$. To find another one, put $\beta = \frac{k_0+1}{1-k_0-k_1} > 1$ and consider the operators acting on the space of positive continuous functions by the following formulae with $u \in C[0; X]$, $X > 0$, and $x \in [0; X]$:

$$\begin{aligned}
Y(u)(x) &= \int_0^x s^\beta u(s)\, ds,\\
P(u)(x) &= p\big(x, Y(u)(x), x^\beta u(x)\big),\\
Q(u)(x) &= Y(u)(x)^{k_0} \cdot (x^\beta u(x))^{k_1} \cdot P(u)(x),\\
F(u)(x) &= x^{-\beta} \int_0^x Q(u)(s)\, ds.
\end{aligned}$$

The last one can be well defined also for $x = 0$ and can be shown to be a contraction. Thus, F has a unique fixed point, i.e. a positive continuous function u on $[0; X]$ such that $F(u) = u$.

Consider the function $y = Y(u)$. According to the definition of the operator Y, we have $y(0) = y'(0) = 0$. Further,

$$y'(x) = x^\beta u(x) = x^\beta F(u)(x) = \int_0^x Q(u)(s)\, ds,$$

whence

$$\begin{aligned}
y''(x) = Q(y)(x) &= Y(u)(x)^{k_0} \cdot (x^\beta u(x))^{k_1} \cdot P(u)(x)\\
&= y(x)^{k_0} y'(x)^{k_1} p\big(x, Y(u)(x), x^\beta u(x)\big) = y(x)^{k_0} y'(x)^{k_1} p(x, y(x), y'(x)).
\end{aligned}$$

So, y is a solution to (1.1), (1.2). It is positive on $(0; X]$ and therefore is just another solution from the statement of the theorem. □

4 Summary

$n = 2$	$0,\, 0$	$Y_0,\, 0$	$0,\, Y_1$	$Y_0,\, Y_1$
$k_0 \geq 1,\, k_1 \geq 1$	U	U	U	U
$k_0 < 1,\, k_1 \geq 1$	U: Th2.3	U	U: Th2.1	U
$k_0 \geq 1,\, k_1 < 1$	U: Th2.3	N: Th2.2	U	U
$k_0 + k_1 \geq 1,\, k_0 < 1,\, k_1 < 1$	U: Th2.3	N: Th2.2	U: Th2.1	U
$k_0 + k_1 < 1$	N: Th2.4	N: Th2.2	U: Th2.1	U

The first column of the above table contains conditions on the positive coefficients k_j. The first row describes initial data, $y(0)$ and $y'(0)$, with Y_0 and Y_1 denoting any non-zero value. In the main part of the table, "U" denotes the uniqueness of solutions to (1.1), (1.2) under the related conditions. "N" denotes non-uniqueness. These labels are followed by references to the related theorems. If not, then the classical existence and uniqueness theorem is implied.

Remark. Asymptotic behavior of unbounded solutions to equation (1.1) with additional conditions

$$0 < p_* \leq p(x,u,v) \leq p^* < \infty, \ \text{ for some } \ p_*, p^* \in \mathbb{R} \ \text{ and all } \ (x,u,v) \in \mathbb{R}^3,$$

is obtained in [4]. Asymptotic behavior of the first derivatives of bounded solutions is described in [5].

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