A Multi-Point Problem for Second Order Differential Equation with Piecewise-Constant Argument of Generalized Type

A. T. Assanova

Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan
E-mails: assanova@math.kz, anartasan@gmail.com

In $[0,T]$ we consider the multi-point problem for the second order differential equation with piecewise-constant argument of generalized type

$$\ddot{x} = a_1(t)\dot{x}(t) + a_2(t)x(t) + a_3(t)\dot{x}(\gamma(t)) + a_4(t)x(\gamma(t)) + f(t), \tag{1}$$

$$\sum_{j=0}^{N} \big\{ b_{1j}\dot{x}(\theta_j) + c_{1j}x(\theta_j) \big\} = d_1, \tag{2}$$

$$\sum_{j=0}^{N} \big\{ b_{2j}\dot{x}(\theta_j) + c_{2j}x(\theta_j) \big\} = d_2, \tag{3}$$

where $x(t)$ is unknown function, the functions $a_i(t)$, $i=\overline{1,4}$ and $f(t)$ are continuous on $[0,T]$; $0=\theta_0<\theta_1<\cdots<\theta_{N-1}<\theta_N=T$, $\theta_j\le\zeta_j\le\theta_{j+1}$ for all $j=0,1,\cdots,N-1$: $\gamma(t)=\zeta_j$ if $t\in[\theta_j,\theta_{j+1})$, $j=\overline{0,N-1}$; b_{sj}, c_{sj} and d_s are constants, where $s=1,2$; $j=\overline{0,N}$.

A solution to problem (1)–(3) is a function $x(t)$, twice continuously differentiable on $[0,T]$, it satisfies equation (1) and the multi-point conditions (2), (3).

The study of differential equations with piecewise-constant argument began with the works by Cook, Busenberg, Wiener, and Shah [11–13, 27, 28]. Many researchers have extensively studied the questions of the existence and uniqueness of solutions, oscillations and stability, integral manifolds and periodic solutions, etc. Differential equations with piecewise-constant argument have been used to develop various models in biology, mechanics, and electronics.

When models are described by differential equations with piecewise-constant argument, the deviation of the argument values is always constant and equal to one, since the greatest integer function is taken as the deviation of the argument. But this approach can contradict real phenomena. In the works by Akhmet [2–4], the greatest integer function as deviating argument was replaced by an arbitrary piecewise constant function. Thus, differential equations with piecewise-constant argument of generalized type are more suitable for modeling and solving various application problems, including areas of neural networks, discontinuous dynamical systems, hybrid systems, etc. To date, the theory of differential equations with piecewise-constant argument of generalized type on the entire axis has been developed and their applications have been implemented. The results were extended to periodic impulse systems of differential equations with piecewise-constant argument of generalized type [5–10]. Along with the study of various properties of differential equations with piecewise-constant argument, a number of authors investigated the questions of solvability and construction of solutions to boundary value problems for these equations on a finite interval [14, 19–23, 25, 26, 29–31]. Particular attention was paid to periodic and multi-point problems for second order differential equations with piecewise-constant argument due to their wide application to natural sciences and engineering [1, 18, 24].

Although the theory of boundary value problems for differential equations with piecewise-constant argument has been developed by a number of researchers, the question of solvability of boundary value problems for systems of differential equations with piecewise-constant argument of generalized type on a finite interval still remains open.

Therefore, the questions of solvability of boundary value problems for such equations are of great importance and relevance. The construction of new general solutions to second order differential equations with piecewise-constant argument of generalized type and investigation into their properties provides an opportunity to solve new classes of problems.

In the present paper, the ideas and results of [15–17] are extended to second order differential equations with piecewise-constant argument of generalized type. We study conditions for unique solvability of multi-point problem for second order differential equation with piecewise-constant argument of generalized type (1)–(3) and construct the algorithms for finding its solution. For this we use the Dzhumabaev parameterization method [15]. The results can be used in the numerical solving of application problems [16].

At first, we introduce new functions $z^{(1)}(t) = x(t)$, $z^{(2)}(t) = \dot{x}(t)$ and rewrite problem (1)–(3) in the following form

$$\dot{z} = A(t)z(t) + A_0(t)z(\gamma(t)) + g(t), \tag{4}$$

$$\sum_{j=0}^{N} C_j z(\theta_j) = d, \tag{5}$$

where $z(t) = col(z^{(1)}(t), z^{(2)}(t))$ is unknown vector function,

$$A(t) = \begin{pmatrix} 0 & 1 \\ a_2(t) & a_1(t) \end{pmatrix}, \quad A_0(t) = \begin{pmatrix} 0 & 0 \\ a_4(t) & a_3(t) \end{pmatrix}, \quad g(t) = \begin{pmatrix} 0 \\ f(t) \end{pmatrix},$$

$$C_j = \begin{pmatrix} c_{1j} & b_{1j} \\ c_{2j} & b_{2j} \end{pmatrix}, \quad j = \overline{0, N}, \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}.$$

A solution to problem (4), (5) is a two-dimensional vector function $z(t)$ which is continuously differentiable on $[0, T]$, it satisfies system (4) and the multi-point condition (5).

Denote by Δ_N a partition of the interval $[0, T)$: $[0, T) = \bigcup\limits_{r=1}^{N} [\theta_{r-1}, \theta_r)$ by lines $t = \theta_j$, $j = \overline{1, N-1}$. Let $z_r(t)$ be a restriction of function $z(t)$ on rth interval $[\theta_{r-1}, \theta_r)$, i.e. $z_r(t) = z(t)$ for $t \in [\theta_{r-1}, \theta_r)$, $r = \overline{1, N}$. Then problem (4), (5) reduce to the following equivalent problem

$$\dot{z}_r = A(t)z_r(t) + A_0(t)z_r(\zeta_{r-1}) + g(t), \ \ t \in [\theta_{r-1}, \theta_r), \ \ r = \overline{1, N}, \tag{6}$$

$$\sum_{j=0}^{N-1} C_j z_{j+1}(\theta_j) + C_N \lim_{t \to T-0} z_N(t) = d, \tag{7}$$

$$\lim_{t \to \theta_p - 0} z_p(t) = z_{p+1}(\theta_p), \ \ p = \overline{1, N-1}. \tag{8}$$

In (4) we take into account that $\gamma(t) = \zeta_j$ if $t \in [\theta_j, \theta_{j+1})$, $j = \overline{0, N-1}$. Condition (8) is the continuity condition of function $z(t)$ on the interior lines $t = t_p$, $p = 0, 1, 2, \dots, N-1$.

Introduce additional parameters $\lambda_r = z_r(\zeta_{r-1})$ for all $r = \overline{1, N}$. On every rth interval we change function $z_r(t)$ by $u_r(t) = z_r(t) - \lambda_r \quad r = \overline{1, N}$.

Then, from (6)–(8), we obtain the following problem with parameters

$$\dot{u}_r = A(t)u_r(t) + [A(t) + A_0(t)]\lambda_r + g(t), \ \ t \in [\theta_{r-1}, \theta_r), \ \ u_r(\zeta_{r-1}) = 0, \ \ r = \overline{1, N}, \tag{9}$$

$$\sum_{j=0}^{N-1} C_j \lambda_{j+1} + \sum_{j=0}^{N-1} C_j u_{j+1}(\theta_j) + C_N \lambda_N + C_N \lim_{t \to T-0} u_N(t) = d, \tag{10}$$

$$\lambda_p + \lim_{t \to \theta_p - 0} z_p(t) = \lambda_{p+1} + z_{p+1}(\theta_p), \ \ p = \overline{1, N-1}. \tag{11}$$

Problems (9) are the Cauchy problems for a system of ordinary differential equations with parameters. Conditions (10), (11) are the relations for determining unknown parameters λ_r, $r=\overline{1,N}$.

Let $X_r(t)$ be a fundamental matrix of differential equation $\dot{u}_r = A(t)u_r(t)$ for $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,N}$. Then, solutions of the Cauchy problems (9) have the following form

$$u_r(t) = X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)[A(\tau)+A_0(\tau)]\,d\tau\lambda_r \\ + X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)g(\tau)\,d\tau,\ \ t\in[\theta_{r-1},\theta_r),\ \ r=\overline{1,N}. \tag{12}$$

Substituting right-hand side of (12) for $t=\theta_j$, $j=\overline{0,N-1}$, $t=T$ to (10), (11), we have

$$\sum_{j=0}^{N-1} C_j[I+D_{j+1}(\theta_j)]\lambda_{j+1} + C_N[I+D_N(T)]\lambda_N = d - \sum_{j=0}^{N-1} C_jF_{j+1}(\theta_j) - C_NF_N(T), \tag{13}$$

$$[I+D_p(\theta_p)]\lambda_p - [I+D_{p+1}(\theta_p)]\lambda_{p+1} = F_{p+1}(\theta_p) - F_p(\theta_p),\ \ p=\overline{1,N-1}, \tag{14}$$

where I is a unit matrix,

$$D_r(t) = X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)[A(\tau)+A_0(\tau)]\,d\tau,$$

$$F_r(t) = X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)g(\tau)\,d\tau,\ \ t\in[\theta_{r-1},\theta_r),\ \ r=\overline{1,N}.$$

We rewrite equations (13), (14) in the following form

$$Q(\Delta_N)\lambda = F(\Delta_N),\ \ \lambda\in R^{2N}. \tag{15}$$

Definition 1. Problem (1)–(3) is called uniquely solvable if, for any triple $(f(t),d_1,d_2)$, where $f(t)\in C([0,T],R)$ and $d_1,d_2\in R$, it has a unique solution.

Theorem 1. *Problem* (1)–(3) *is solvable if and only if the vector* $F(\Delta_N)$ *is orthogonal to the kernel of the transposed matrix* $(Q(\Delta_N))'$, *i.e., for any* $\xi\in Ker(Q(\Delta_N))'$, *the following equality is true:* $(F(\Delta_N),\xi)=0$, *where* $(\cdot,\cdot)$ *is the scalar product in* R^{2N}.

Theorem 2. *Problem* (1)–(3) *is uniquely solvable if and only if the* $(2N\times 2N)$ *matrix* $Q(\Delta_N)$ *is invertible.*

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