On the Well-Posed Criterion of General Linear Boundary Value Problems for Systems of Linear Impulsive Differential Equations with Infinity Points of Impulse Actions

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In the presentation, we consider the well-posed question for the general linear boundary value problem for the impulsive differential systems

$$\frac{dx}{dt}=P_0(t)x+q_0(t)\ \text{ for a.a. }\ t\in I\setminus T, \tag{1}$$

$$x(\tau_l+)-x(\tau_l-)=G_0(\tau_l)x(\tau_l)+u_0(\tau_l)\ \ (l=1,2,\dots); \tag{2}$$

$$\ell_0(x)=c_0, \tag{3}$$

where $I=[a,b]\subset\mathbb{R}$, $P_0\in L(I;\mathbb{R}^{n\times n})$, $q_0\in L(I;\mathbb{R}^n)$, $G_0\in B(T;\mathbb{R}^{n\times n})$, $u_0\in B(T;\mathbb{R}^n)$, $T=\{\tau_1,\tau_2,\dots\}$, $\tau_l\in I$ $(l=1,2,\dots)$, $\tau_l\neq\tau_k$ if $l\neq k$ $(l,k=1,2,\dots)$, $\ell_0:\mathrm{BV}(I;\mathbb{R}^n)\to\mathbb{R}^n$ is a linear vector-functional, bounded with respect to the norm $\|.\|_\infty$, and $c_0\in\mathbb{R}^n$.

Along with the impulsive general boundary (1)–(3), consider the sequence of problems

$$\frac{dx}{dt}=P_m(t)x+q_m(t)\ \text{ for a.a. }\ t\in I\setminus T, \tag{1_m}$$

$$x(\tau_l+)-x(\tau_{l-})=G_m(\tau_l)x(\tau_l)+u_m(\tau_l)\ \ (l=1,2,\dots); \tag{2_m}$$

$$\ell_m(x)=c_m \tag{3_m}$$

$(m=1,2,\dots)$, where $P_m\in L(I;\mathbb{R}^{n\times n})$, $q_m\in L(I;\mathbb{R}^n)$, $G_m\in B(T;\mathbb{R}^{n\times n})$, $u_m\in B(T;\mathbb{R}^n)$, $\ell_m:\mathrm{BV}(I;\mathbb{R}^n)\to\mathbb{R}^n$ is a linear vector-functional, bounded with respect to the norm $\|\cdot\|_\infty$, and $c_m\in\mathbb{R}^n$ $(m=1,2,\dots)$.

We give the necessary and sufficient conditions (as well, some effective sufficient conditions) for the existence of a unique solution for problem (1_m)–(3_m) for every sufficiently large m and the nearness these solutions to the solution of problem (1)–(3). The problem quite fully is already investigated in [3] (see also the references therein). Such problem was studied in [3–5] for linear ordinary differential systems.

Similar problem is investigated in [2] (see also the references therein) for the initial problems for linear impulsive systems.

A number of issues of the theory of linear systems of differential equations with impulsive effect have been studied sufficiently well [1–3, 6] (see also the references therein).

The use will be made of the following notation and definitions.

$\mathbb{R}=]-\infty,+\infty[$. $\mathbb{R}^{n\times m}$ is the space of all real $n\times m$ matrices $X=(x_{i,j})_{i,j=1}^{n,m}$ with the norm $\|X\|=\max\limits_{j=1,\dots,m}\sum\limits_{i=1}^{n}|x_{ij}|$. I_n is the identity $n\times n$-matrix.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t.

$\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of the components of the matrix-function X.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the space of all bounded variation matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$, with the norm $\|X\|_\infty = \sup\{\|X(t)\| : \ t \in [a,b]\}$.

$\mathrm{AC}([a,b];\mathbb{R}^{n\times m})$ is the set of all absolutely continuous matrix-functions.

$\mathrm{AC}_{loc}(J;\mathbb{R}^{n\times m})$, where $J \subset \mathbb{R}$, is the set of all matrix-functions whose restrictions to an arbitrary closed interval $[a,b]$ from J belong to $\mathrm{AC}([a,b];D)$.

$\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times m}) = BV(I;\mathbb{R}^{n\times m}) \cap \mathrm{AC}_{loc}(I \setminus T;\mathbb{R}^{n\times m})$.

$B(T;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G : T \to \mathbb{R}^{n\times m}$ such that $\sum\limits_{l=1}^{+\infty} \|G(\tau_l)\| < +\infty$;

$|||\ell|||$ is the norm of a linear bounded vector-functional ℓ.

For the corresponding matrix-functions X, Y and Z, we set

$$\mathcal{B}_\iota(X;Y,Z)(t) \equiv \int_a^t X(\tau)Y(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} X(\tau_l+)\,Z(\tau_l).$$

Everywhere, we assume that

$$\lim_{m\to+\infty} \ell_m(x) = \ell_0(x) \ \text{ for } \ x \in \mathrm{BV}(I;\mathbb{R}^n), \ \ \limsup_{m\to+\infty} |||\ell_m||| < +\infty$$

and $\det(I_n + G(\tau_l)) \neq 0$ $(l = 1,2,\dots)$.

The last inequalities guarantee the unique solvability of the Cauchy problem for the impulsive system (1), (2) (see [2, 6]).

Definition 1. A vector-function $x \in \mathrm{AC}_{loc}(I \setminus T;\mathbb{R}^n)$ is said to be a solution of system (1), (2) if $x'(t) = P(t)x(t) + q(t)$ for a.a. $t \in I \setminus T$ and there exist onesided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1,2,\dots)$ satisfying equalities (2).

Without loss of generality, we can assume that the solution x of the impulsive differential system (1), (2) is continuous from the left at the points of the impulses actions τ_l $(l = 1,2,\dots)$, i.e., $x(\tau_l) = x(\tau_l-)$ $(l = 1,2,\dots)$.

Let x_0 be a unique solution of problem (1)–(3) (about existence conditions see, for example, [1, 3, 6]).

We give the necessary and sufficient and effective sufficient conditions for the boundary value problem (1_m)–(3_m) to have a unique solution x_m for any sufficiently large m and

$$\lim_{m\to+\infty} \|x_m - x_0\|_\infty = 0. \tag{4}$$

Remark 1. If we consider the case where for every natural m, the impulses points depend on m in the impulsive systems $(1_m), (2_m)$ $(m = 1,2,\dots)$, in particular, the linear algebraic system (2_m) has the form

$$x(\tau_{lm}+) - x(\tau_{lm}-) = G_m(\tau_{lm})x(\tau_{lm}) + u_m(\tau_{lm}) \ \ (l = 1,2,\dots),$$

where $\tau_{lm} \in I$ $(l = 1,2,\dots)$, then the last general case will be reduced to case (2_m) using the following conception given in [2, 3].

Along with systems $(1), (2)$ and $(1_m), (2_m)$ $(m = 1, 2, \dots)$, we consider the corresponding homogeneous systems

$$\frac{dx}{dt} = P_m(t)x \ \text{ for a.a. } \ t \in I \setminus T, \tag{1_{m0}}$$

$$x(\tau_l+) - x(\tau_l-) = G_m(\tau_l)x(\tau_l) \ \ (l = 1, 2, \dots). \tag{2_{m0}}$$

Definition 2. We say that the sequence $(P_m, q_m; G_m, u_m; \ell_m)$ $(m = 1, 2, \dots)$ belongs to the set $\mathcal{S}(P_0, q_0; G_0, u_0; \ell_0)$ if for every $c_0 \in \mathbb{R}^n$ and $c_m \in \mathbb{R}^n$ $(m = 1, 2, \dots)$, satisfying condition $\lim\limits_{k\to+\infty} c_m = c_0$, problem (1_m)–(3_m) has a unique solution x_m for any sufficiently large m and condition (4) holds.

Theorem 1. *The inclusion*

$$\big((P_m, q_m; G_m, u_m; \ell_m)\big)_{m=1}^{\infty} \in \mathcal{S}(P_0, q_0; G_0, u_0; \ell_0) \tag{5}$$

holds if and only if there exists a sequence $H_m \in \mathrm{BVAC}_{loc}(I, T; \mathbb{R}^{n\times n})$ $(m = 0, 1, \dots)$ *such that condition*

$$\limsup_{m\to+\infty} \bigvee_a^b \big(H_m + \mathcal{B}_\iota(H_m; P_m, G_m)\big) < +\infty \tag{6}$$

holds, and conditions

$$\lim_{m\to+\infty} H_m(t) = I_n, \tag{7}$$

$$\lim_{m\to+\infty} \mathcal{B}_\iota(H_m; P_m, G_m)(t) = \mathcal{B}_\iota(I_n; P_0, G_0)(t),$$

$$\lim_{m\to+\infty} \mathcal{B}_\iota(H_m; q_m, u_m)(t) = \mathcal{B}_\iota(I_n; q_0, u_0)(t)$$

hold uniformly on I*.*

Theorem 2. *Let* $\det(I_n + G_m(\tau_l)) \neq 0$ $(l = 1, 2, \dots;\ m = 0, 1, \dots)$*. Then inclusion* (5) *holds if and only if the conditions*

$$\lim_{m\to+\infty} X_m^{-1}(t) = I_n,$$

$$\lim_{m\to+\infty} \bigg(\int_a^t X_m^{-1}(\tau) q_m(\tau)\,d\tau + \sum_{\tau_l \in [a,t[} X_m^{-1}(\tau_l+)\,u_m(\tau_l)\bigg) = \int_a^t q_0(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} u_0(\tau_l)$$

hold uniformly on I*, where* X_m *is the fundamental matrix of the homogeneous system* $(1_{m0}), (2_{m0})$ $(m = 1, 2, \dots)$.

Remark 2. Note that condition (6) holds if

$$\limsup_{m\to+\infty} \bigg(\int_a^b \big\|H_m'(t) + H_m(t)P_m(t)\big\|\,dt + \sum_{l=1}^{+\infty} \|d_2 H_m(\tau_l) + H_m(\tau_l+)G_m(\tau_l)\|\bigg) < +\infty.$$

Now we give some effective sufficient conditions guaranteeing inclusion (5).

Theorem 3. *Let the condition*

$$\limsup_{m\to+\infty}\bigg(\int\limits_a^b \|P_m(t)\|\,dt+\sum_{l=1}^{\infty}\|G_m(\tau_l)\|\bigg)<+\infty$$

hold and let the conditions

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_m(\tau_l)\bigg)=\int\limits_a^t P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_0(\tau_l),$$
$$\lim_{m\to+\infty}\bigg(\int\limits_a^t q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_m(\tau_l)\bigg)=\int\limits_m^t q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0(\tau_l)$$

hold uniformly on I. Then inclusion (5) *holds.*

Corollary 1. *Let* (6) *hold and let conditions* (7),

$$\lim_{m\to+\infty}\int\limits_a^t H_m(\tau)P_m(\tau)d\tau=\int\limits_a^t P_0(\tau)d\tau,\ \lim_{m\to+\infty}\int\limits_a^t H_m(\tau)q_m(\tau)d\tau=\int\limits_a^t q_0(\tau)\,d\tau$$

hold uniformly on I, and the conditions

$$\lim_{m\to+\infty}G_m(\tau_l)=G_0(\tau_l)\ \text{ and }\ \lim_{m\to+\infty}u_m(\tau_l)=u_0(\tau_l)$$

hold uniformly on T, where $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$. Let, moreover, either

$$\limsup_{m\to+\infty}\sum_{l=1}^{\infty}\big(\|G_m(\tau_l)\|+\|u_m(\tau_l)\|\big)<+\infty\ \text{ or }\ \limsup_{m\to+\infty}\sum_{l=1}^{\infty}\|H_m(\tau_l+)-H_m(\tau_l)\|<+\infty.$$

Then inclusion (5) *holds.*

Corollary 2. *Let condition* (6) *hold and let the conditions*

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_m(\tau_l)\bigg)=B(t)-B(a),$$
$$\lim_{m\to+\infty}\bigg(\int\limits_a^t H_m(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}(B(\tau_l+)-G_m(\tau_l+))\,G_m(\tau_l)\bigg)=\int\limits_a^t P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_0(\tau_l),$$
$$\lim_{m\to+\infty}\bigg(\int\limits_a^t H_m(\tau)\,q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}(B(\tau_l+)-G_m(\tau_l+))\,u_m(\tau_l)\bigg)=\int\limits_{t_0}^t q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0(\tau_l)$$

hold uniformly on I, where $B\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ and

$$H_m(t)\equiv I_n-\int\limits_a^t P_m(\tau)\,d\tau-\sum_{\tau_l\in[a,t[}G_m(\tau_l)+B(t)-B(a)\ \ (m=1,2,\dots).$$

Then inclusion (5) *holds.*

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