On Well-Possedness of General Linear Boundary Value Problems for High Order Ordinary Linear Differential Equations

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We consider the question on the well-posedness of the boundary value problem

$$u^{(n)}=\sum_{l=1}^{n}p_l(t)u^{(l-1)}+p_0(t)\ \text{ for a.a. } t\in I, \tag{1}$$

$$\ell_i(u,u',\dots,u^{(n-1)})=c_{i0}\ \ (i=1,\dots,n), \tag{2}$$

where $I=[a,b]$ is an arbitrary closed interval from $\mathbb{R}$, $p_l\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $c_{io}\in\mathbb{R}$ $(i=1,\dots,n)$, and $\ell_i:\mathrm{AC}^{(n-1)}(I;\mathbb{R})\to\mathbb{R}$ $(i=1,\dots,n)$ are linear bounded functionals with respect to the norm

$$\|u\|_{\mathrm{AC}}=\sum_{j=1}^{n}\|u^{(j-1)}\|_c.$$

Here $\mathrm{AC}^{(n-1)}(I;\mathbb{R})$ is the set of all functions $u:I\to\mathbb{R}$ such that the derivatives $u^{(j)}$ $(j=0,\dots,n-1)$ are absolutely continuous functions on I, i.e., such that $u^{(j)}\in\mathrm{AC}(I;\mathbb{R})$ $(j=0,\dots,n-1)$, and $\|v\|_c=\max\{|v(t)|:t\in I\}$ for every continuous function $v:I\to\mathbb{R}$.

By $|||\ell|||$ we denote the usual norm of the linear operator ℓ.

Under a solution of the differential equation (1) we understand a function $u\in\mathrm{AC}^{(n-1)}(I;\mathbb{R})$ such that

$$u^{(n)}(t)=\sum_{l=1}^{n}p_l(t)u^{(l-1)}(t)+p_0(t)\ \text{ for a.a. } t\in I.$$

Let u_0 be the unique solution of the Cauchy problem $(1),(2)$.

Along with problem $(1),(2)$ consider the sequence of problems

$$u^{(n)}=\sum_{l=1}^{n}p_{lk}(t)u^{(l-1)}+p_{0k}(t)\ \text{ for a.a. } t\in I, \tag{1_k}$$

$$\ell_{ik}(u,u',\dots,u^{(n-1)})=c_{ik}\ \ (i=1,\dots,n), \tag{2_k}$$

$(k=1,2,\dots)$, where $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $c_{ik}\in\mathbb{R}$ $(i=1,\dots,n;\ k=1,2,\dots)$, and $\ell_{ik}:\mathrm{AC}^{(n-1)}(I;\mathbb{R})\to\mathbb{R}$ $(i=1,\dots,n;\ k=1,2,\dots)$ are linear bounded functionals.

Definition. We say that the sequence $(p_{1k},\dots,p_{nk},p_{0k};\ell_{1k},\dots,\ell_{nk})$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(p_1,\dots,p_n,p_0;\ell_1,\dots,\ell_n)$ if for every $c_{i0}\in\mathbb{R}$ $(i=1,\dots,n)$ and a sequence $c_{ik}\in\mathbb{R}$ $(i=1,\dots,n;\ k=1,2,\dots)$, satisfying the condition

$$\lim_{k\to+\infty} c_{ik}=c_{i0}\ \ (i=1,\dots,n), \tag{3}$$

the boundary value problem $(1_k),(2_k)$ has the unique solution u_k for any natural k and

$$\lim_{k\to+\infty} u_k^{(i-1)}(t)=u_0^{(i-1)}(t)\ \ (i=1,\dots,n) \tag{4}$$

uniformly on I.

Along with equations (1) and (1_k) $(k=1,2,\dots)$ we consider the corresponding homogeneous equations

$$u^{(n)}=\sum_{l=1}^{n} p_l(t)u^{(i-1)}\ \text{ for a.a }\ t\in I \tag{1_0}$$

and

$$u^{(n)}=\sum_{l=1}^{n} p_{lk}(t)u^{(i-1)}\ \text{ for a.a }\ t\in I \tag{1_{0k}}$$

$(k=1,2,\dots)$.

If the functions $v_i\in \mathrm{AC}^{(n-1)}(I;\mathbb{R})$ $(i=1,\dots,n)$, then by

$$w_0(v_1,\dots,v_n)(t)=\det\big((v_i^{(l-1)}(t))_{i,l=1}^n\big)$$

we denote the so called Wronskiĭ's determinant, and by $w_{il}(v_1,\dots,v_n)(t)$ $(i,l=1,\dots,n)$ we denote the cofactor of the il-element of $w_0(v_1,\dots,v_n)$.

Let u_l $(l=1,\dots,n)$ and u_{lk} $(l=1,\dots,n;\ k=1,2,\dots)$ be the fundamental systems of solutions of the homogeneous systems (1_0) and (2_{0k}) $(k=1,2,\dots)$, respectively.

Below we give necessary and sufficient conditions, as well some sufficient conditions, guaranteeing the inclusion

$$\big((p_{1k},\dots,p_{nk},p_{0k};\ell_{1k},\dots,\ell_{nk})\big)_{k=1}^{+\infty}\in\mathcal{S}(p_1,\dots,p_n,p_0;\ell_1,\dots,\ell_n). \tag{5}$$

Theorem 1. *Let the functions $p_l\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and let the linear functionals ℓ_i, ℓ_{ik} $(i=1,\dots,n;\ k=1,2,\dots)$ be such that the conditions*

$$\lim_{k\to+\infty}\ell_{ik}(u,u',\dots,u^{(n-1)})=\ell_i(u,u',\dots,u^{(n-1)})\ \ \textit{for}\ \ u\in\mathrm{AC}^{(n-1)}(I;\mathbb{R})\ \ (i=1,\dots,n), \tag{6}$$

$$\limsup_{k\to+\infty}|||\ell_{ik}|||<+\infty\ \ (i=1,\dots,n) \tag{7}$$

hold. Then inclusion (5) *holds if and only if there exists a sequence of functions h_{il}, $h_{ilk}\in\mathrm{AC}(I;\mathbb{R})$ $(i,l=1,\dots,n;\ k=1,2,\dots)$ such that the conditions*

$$\inf\Big\{\big|\det((h_{il}(t))_{i,l=1}^n)\big|:\ t\in I\Big\}>0 \tag{8}$$

and

$$\limsup_{k\to+\infty}\sum_{i,l=1}^{n}\int_a^b\Big|h'_{ilk}(t)+h_{i\,l-1\,k}(t)\operatorname{sgn}(l-1)+h_{ink}(t)\,p_l(t)\Big|\,dt<+\infty \tag{9}$$

hold, and the conditions

$$\lim_{k\to+\infty} h_{ilk}(t) = h_{il}(t) \ \ (i,l = 1,\dots,n) \tag{10}$$

and

$$\lim_{k\to+\infty} \int\limits_a^t h_{ink}(\tau)p_{lk}(\tau)\,d\tau = \int\limits_a^t h_{in}(\tau)p_l(\tau)\,d\tau \ \ (i = 1,\dots,n;\ l = 0,\dots,n)$$

hold uniformly on I.

Theorem 2. *Let the functions $p_l \in L(I;\mathbb{R})$ $(l = 0,\dots,n)$, $p_{lk} \in L(I;\mathbb{R})$ $(l = 0,\dots,n;\ k = 1,2,\dots)$ and let the linear functionals ℓ_i, ℓ_{ik} $(i = 1,\dots,n;\ k = 1,2,\dots)$ be such that conditions* (6) *and* (7) *hold. Then inclusion* (5) *holds if and only if the conditions*

$$\lim_{k\to+\infty} u_{lk}^{(i-1)}(t) = u_l^{(i-1)}(t) \ \ (i,l = 1,\dots,n)$$

and

$$\lim_{k\to+\infty} \int\limits_a^t \frac{w_{in}(u_{1k},\dots,u_{nk})(\tau)}{w_0(u_{1k},\dots,u_{nk})(\tau)}\,p_{0k}(\tau)\,d\tau = \int\limits_a^t \frac{w_{in}(u_1,\dots,u_n)(\tau)}{w_0(u_1,\dots,u_n)(\tau)}\,p_0(\tau)\,d\tau \ \ (i = 1,\dots,n) \tag{11}$$

hold uniformly on I.

Theorem 3. *Let the functions $p_l \in L(I;\mathbb{R})$ $(l = 0,\dots,n)$, $p_{lk} \in L(I;\mathbb{R})$ $(l = 0,\dots,n;\ k = 1,2,\dots)$ and let the linear functionals ℓ_i, ℓ_{ik} $(i = 1,\dots,n;\ k = 1,2,\dots)$ be such that conditions* (6), (7) *and*

$$\limsup_{k\to+\infty} \int\limits_a^b \|p_{lk}(t)\|\,dt < +\infty \ \ (l = 1,\dots,n)$$

hold, and the condition

$$\lim_{k\to+\infty} \int\limits_a^t p_{lk}(\tau)\,d\tau = \int\limits_a^t p_l(\tau)\,d\tau \ \ (l = 0,\dots,n)$$

hold uniformly on I. Then the boundary value problem $(1_k), (2_k)$ has the unique solution u_k for any natural k and condition (4) *holds uniformly on I.*

Corollary 1. *Let the functions $p_l \in L(I;\mathbb{R})$ $(l = 0,\dots,n)$, $p_{lk} \in L(I;\mathbb{R})$ $(l = 0,\dots,n;\ k = 1,2,\dots)$ and let the linear functionals ℓ_i, ℓ_{ik} $(i = 1,\dots,n;\ k = 1,2,\dots)$ be such that conditions* (3), (6), (7) *and* (9) *hold, and conditions* (10) *and*

$$\lim_{k\to+\infty} \int\limits_a^t h_{ink}(\tau)p_{lk}(\tau)\,d\tau = \int\limits_a^t p_l^*(\tau)\,d\tau \ \ (i = 1,\dots,n;\ l = 0,\dots,n)$$

hold uniformly on I, where $p_l^* \in L(I;\mathbb{R})$ $(l = 0,\dots,n)$; h_{il}, $h_{ilk} \in \mathrm{AC}(I;\mathbb{R})$ $(i,l = 1,\dots,n;$ $k = 1,2,\dots)$. Then the inclusion

$$\big((p_{1k},\dots,p_{nk},p_{0k};\ell_{1k},\dots,\ell_{nk})\big)_{k=1}^{+\infty} \in \mathcal{S}(p_1 - p_1^*,\dots,p_n - p_n^*,p_0 - p_0^*;\ell_1,\dots,\ell_n)$$

holds.

Remark. In Theorem 2 and Corollary 1, without loss of generality we can assume that $h_{ii}(t) \equiv 1$ and $h_{il}(t) \equiv 0$ $(i \neq l;\ i, l = 1, \dots, n)$. So condition (8) is valid evidently.

Remark. If $n = 2$ in Theorem 3, then condition (11) has the form

$$\lim_{k\to+\infty} \int_a^t \frac{u'_{1k}(\tau)p_{0k}(\tau)}{u_{1k}(\tau)u'_{2k}(\tau) - u_{2k}(\tau)u'_{1k}(\tau)}\,d\tau = \int_a^t \frac{u'_1(\tau)p_0(\tau)}{u_1(\tau)u'_2(\tau) - u_2(\tau)u'_1(\tau)}\,d\tau,$$
$$\lim_{k\to+\infty} \int_a^t \frac{u_{1k}(\tau)p_{0k}(\tau)}{u_{1k}(\tau)u'_{2k}(\tau) - u_{2k}(\tau)u'_{1k}(\tau)}\,d\tau = \int_a^t \frac{u_1(\tau)p_0(\tau)}{u_1(\tau)u'_2(\tau) - u_2(\tau)u'_1(\tau)}\,d\tau.$$

In the equalities we can take u_{2k} instead of u_{1k} $(k = 1, 2, \dots)$ and u_2 instead of u_1.

For the proof we use the well-know concept. It is well known that if the function u is a solution of problem (1), (2), then the vector-function $x = (x_i)_{i=1}^n$, $x_i = u^{(i-1)}$ $(i = 1, \dots, n)$ is a solution of the following general linear boundary value problem for system of ordinary differential equations

$$\frac{dx}{dt} = P(t)\,x + q(t),$$
$$\ell(x) = c_0,$$

where the matrix- and vector-functions $P(t) = (p_{il}(t))_{i,l=1}^n$ and $q(t) = (q_i(t))_{i=1}^n$ are defined, respectively, by

$$p_{il}(t) \equiv 0,\ \ p_{i\,i+1} \equiv 1\ \ (l \neq i+1;\ i = 1, \dots, n-1;\ l = 1, \dots, n),$$
$$p_{nl}(t) \equiv p_l(t)\ \ (l = 1, \dots, n);$$
$$q_i(t) \equiv 0\ \ (i = 1, \dots, n-1),\ \ q_n(t) \equiv p_0(t);$$
$$\ell(x) = \big(\ell_l(u, u', \dots, u^{(n-1)})\big)_{l=1}^n\ \ (x = (u^{(l-1)})_{l=1}^n);\ \ c_0 = (c_{l0})_{l=1}^n.$$

Analogously, problem $(1_k), (2_k)$ can be rewritten in the form of the last type problem for every natural k. So, using the results contained in [1–3] we get the results given above.

References

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