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ABSTRACTS

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On the Well-Posed Criterion of General Linear Boundary Value Problems for Systems of Linear Impulsive Differential Equations with Infinity Points of Impulse Actions

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In the presentation, we consider the well-posed question for the general linear boundary value problem for the impulsive differential systems

$$\frac{dx}{dt} = P_0(t)x + q_0(t) \ \text{ for a.a. } \ t \in I \setminus T, \tag{1}$$

$$x(\tau_l+) - x(\tau_l-) = G_0(\tau_l)x(\tau_l) + u_0(\tau_l) \ \ (l = 1, 2, \dots); \tag{2}$$

$$\ell_0(x) = c_0, \tag{3}$$

where $I = [a, b] \subset \mathbb{R}$, $P_0 \in L(I; \mathbb{R}^{n\times n})$, $q_0 \in L(I; \mathbb{R}^n)$, $G_0 \in B(T; \mathbb{R}^{n\times n})$, $u_0 \in B(T; \mathbb{R}^n)$, $T = \{\tau_1, \tau_2, \dots\}$, $\tau_l \in I$ $(l = 1, 2, \dots)$, $\tau_l \neq \tau_k$ if $l \neq k$ $(l, k = 1, 2, \dots)$, $\ell_0 : \mathrm{BV}(I; \mathbb{R}^n) \to \mathbb{R}^n$ is a linear vector-functional, bounded with respect to the norm $\|.\|_\infty$, and $c_0 \in \mathbb{R}^n$.

Along with the impulsive general boundary (1)–(3), consider the sequence of problems

$$\frac{dx}{dt} = P_m(t)x + q_m(t) \ \text{ for a.a. } \ t \in I \setminus T, \tag{1_m}$$

$$x(\tau_l+) - x(\tau_{l-}) = G_m(\tau_l)x(\tau_l) + u_m(\tau_l) \ \ (l = 1, 2, \dots); \tag{2_m}$$

$$\ell_m(x) = c_m \tag{3_m}$$

$(m = 1, 2, \dots)$, where $P_m \in L(I; \mathbb{R}^{n\times n})$, $q_m \in L(I; \mathbb{R}^n)$, $G_m \in B(T; \mathbb{R}^{n\times n})$, $u_m \in B(T; \mathbb{R}^n)$, $\ell_m : \mathrm{BV}(I; \mathbb{R}^n) \to \mathbb{R}^n$ is a linear vector-functional, bounded with respect to the norm $\|\cdot\|_\infty$, and $c_m \in \mathbb{R}^n$ $(m = 1, 2, \dots)$.

We give the necessary and sufficient conditions (as well, some effective sufficient conditions) for the existence of a unique solution for problem (1_m)–(3_m) for every sufficiently large m and the nearness these solutions to the solution of problem (1)–(3). The problem quite fully is already investigated in [3] (see also the references therein). Such problem was studied in [3–5] for linear ordinary differential systems.

Similar problem is investigated in [2] (see also the references therein) for the initial problems for linear impulsive systems.

A number of issues of the theory of linear systems of differential equations with impulsive effect have been studied sufficiently well [1–3, 6] (see also the references therein).

The use will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$. $\mathbb{R}^{n\times m}$ is the space of all real $n \times m$ matrices $X = (x_{i,j})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\limits_{j=1,\dots,m} \sum\limits_{i=1}^{n} |x_{ij}|$. I_n is the identity $n \times n$-matrix.

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^n$.

$X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of the matrix-function $X : [a,b] \to \mathbb{R}^{n\times m}$ at the point t.

$\bigvee\limits_a^b(X)$ is the sum of total variations on $[a,b]$ of the components of the matrix-function X.

$\mathrm{BV}([a,b];\mathbb{R}^{n\times m})$ is the space of all bounded variation matrix-functions $X : [a,b] \to \mathbb{R}^{n\times m}$, with the norm $\|X\|_\infty = \sup\{\|X(t)\| : \ t \in [a,b]\}$.

$\mathrm{AC}([a,b];\mathbb{R}^{n\times m})$ is the set of all absolutely continuous matrix-functions.

$\mathrm{AC}_{loc}(J;\mathbb{R}^{n\times m})$, where $J \subset \mathbb{R}$, is the set of all matrix-functions whose restrictions to an arbitrary closed interval $[a,b]$ from J belong to $\mathrm{AC}([a,b];D)$.

$\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times m}) = BV(I;\mathbb{R}^{n\times m}) \cap \mathrm{AC}_{loc}(I\setminus T;\mathbb{R}^{n\times m})$.

$B(T;\mathbb{R}^{n\times m})$ is the set of all matrix-functions $G : T \to \mathbb{R}^{n\times m}$ such that $\sum\limits_{l=1}^{+\infty} \|G(\tau_l)\| < +\infty$;

$|||\ell|||$ is the norm of a linear bounded vector-functional ℓ.

For the corresponding matrix-functions X, Y and Z, we set

$$\mathcal{B}_\iota(X;Y,Z)(t) \equiv \int\limits_a^t X(\tau)Y(\tau)\,d\tau + \sum_{\tau_l\in[a,t[} X(\tau_l+)\,Z(\tau_l).$$

Everywhere, we assume that

$$\lim_{m\to+\infty} \ell_m(x) = \ell_0(x) \ \text{ for } \ x \in \mathrm{BV}(I;\mathbb{R}^n), \ \ \limsup_{m\to+\infty} |||\ell_m||| < +\infty$$

and $\det(I_n + G(\tau_l)) \neq 0$ $(l = 1,2,\dots)$.

The last inequalities guarantee the unique solvability of the Cauchy problem for the impulsive system (1), (2) (see [2,6]).

Definition 1. A vector-function $x \in \mathrm{AC}_{loc}(I\setminus T;\mathbb{R}^n)$ is said to be a solution of system (1), (2) if $x'(t) = P(t)x(t) + q(t)$ for a.a. $t \in I \setminus T$ and there exist onesided limits $x(\tau_l-)$ and $x(\tau_l+)$ $(l = 1,2,\dots)$ satisfying equalities (2).

Without loss of generality, we can assume that the solution x of the impulsive differential system (1), (2) is continuous from the left at the points of the impulses actions τ_l $(l = 1,2,\dots)$, i.e., $x(\tau_l) = x(\tau_l-)$ $(l = 1,2,\dots)$.

Let x_0 be a unique solution of problem (1)–(3) (about existence conditions see, for example, [1,3,6]).

We give the necessary and sufficient and effective sufficient conditions for the boundary value problem (1_m)–(3_m) to have a unique solution x_m for any sufficiently large m and

$$\lim_{m\to+\infty} \|x_m - x_0\|_\infty = 0. \tag{4}$$

Remark 1. If we consider the case where for every natural m, the impulses points depend on m in the impulsive systems $(1_m), (2_m)$ $(m = 1,2,\dots)$, in particular, the linear algebraic system (2_m) has the form

$$x(\tau_{lm}+) - x(\tau_{lm}-) = G_m(\tau_{lm})x(\tau_{lm}) + u_m(\tau_{lm}) \ \ (l = 1,2,\dots),$$

where $\tau_{lm} \in I$ $(l = 1,2,\dots)$, then the last general case will be reduced to case (2_m) using the following conception given in [2,3].

Along with systems $(1),(2)$ and $(1_m),(2_m)$ $(m=1,2,\dots)$, we consider the corresponding homogeneous systems

$$\frac{dx}{dt}=P_m(t)x \ \text{ for a.a. } t\in I\setminus T, \tag{1_{m0}}$$

$$x(\tau_l+)-x(\tau_l-)=G_m(\tau_l)x(\tau_l) \ \ (l=1,2,\dots). \tag{2_{m0}}$$

Definition 2. We say that the sequence $(P_m,q_m;G_m,u_m;\ell_m)$ $(m=1,2,\dots)$ belongs to the set $\mathcal{S}(P_0,q_0;G_0,u_0;\ell_0)$ if for every $c_0\in\mathbb{R}^n$ and $c_m\in\mathbb{R}^n$ $(m=1,2,\dots)$, satisfying condition $\lim\limits_{k\to+\infty}c_m=c_0$, problem (1_m)–(3_m) has a unique solution x_m for any sufficiently large m and condition (4) holds.

Theorem 1. *The inclusion*

$$\big((P_m,q_m;G_m,u_m;\ell_m)\big)_{m=1}^{\infty}\in\mathcal{S}(P_0,q_0;G_0,u_0;\ell_0) \tag{5}$$

holds if and only if there exists a sequence $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=0,1,\dots)$ such that condition

$$\limsup_{m\to+\infty}\bigvee_a^b\big(H_m+\mathcal{B}_\iota(H_m;P_m,G_m)\big)<+\infty \tag{6}$$

holds, and conditions

$$\begin{gathered}\lim_{m\to+\infty}H_m(t)=I_n,\\ \lim_{m\to+\infty}\mathcal{B}_\iota(H_m;P_m,G_m)(t)=\mathcal{B}_\iota(I_n;P_0,G_0)(t),\\ \lim_{m\to+\infty}\mathcal{B}_\iota(H_m;q_m,u_m)(t)=\mathcal{B}_\iota(I_n;q_0,u_0)(t)\end{gathered} \tag{7}$$

hold uniformly on I.

Theorem 2. *Let $\det(I_n+G_m(\tau_l))\neq0$ $(l=1,2,\dots;\ m=0,1,\dots)$. Then inclusion* (5) *holds if and only if the conditions*

$$\begin{gathered}\lim_{m\to+\infty}X_m^{-1}(t)=I_n,\\ \lim_{m\to+\infty}\bigg(\int_a^t X_m^{-1}(\tau)q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}X_m^{-1}(\tau_l+)\,u_m(\tau_l)\bigg)=\int_a^t q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0(\tau_l)\end{gathered}$$

hold uniformly on I, where X_m is the fundamental matrix of the homogeneous system $(1_{m0}),(2_{m0})$ $(m=1,2,\dots)$.

Remark 2. Note that condition (6) holds if

$$\limsup_{m\to+\infty}\bigg(\int_a^b\big\|H_m'(t)+H_m(t)P_m(t)\big\|\,dt+\sum_{l=1}^{+\infty}\|d_2H_m(\tau_l)+H_m(\tau_l+)G_m(\tau_l)\|\bigg)<+\infty.$$

Now we give some effective sufficient conditions guaranteeing inclusion (5).

Theorem 3. *Let the condition*

$$\limsup_{m\to+\infty}\bigg(\int\limits_a^b \|P_m(t)\|\,dt+\sum_{l=1}^{\infty}\|G_m(\tau_l)\|\bigg)<+\infty$$

hold and let the conditions

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_m(\tau_l)\bigg)=\int\limits_a^t P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_0(\tau_l),$$

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_m(\tau_l)\bigg)=\int\limits_m^t q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0(\tau_l)$$

hold uniformly on I. Then inclusion (5) *holds.*

Corollary 1. *Let* (6) *hold and let conditions* (7)*,*

$$\lim_{m\to+\infty}\int\limits_a^t H_m(\tau)P_m(\tau)d\tau=\int\limits_a^t P_0(\tau)d\tau,\ \lim_{m\to+\infty}\int\limits_a^t H_m(\tau)q_m(\tau)d\tau=\int\limits_a^t q_0(\tau)\,d\tau$$

hold uniformly on I, and the conditions

$$\lim_{m\to+\infty}G_m(\tau_l)=G_0(\tau_l)\ \text{ and }\ \lim_{m\to+\infty}u_m(\tau_l)=u_0(\tau_l)$$

hold uniformly on T, where $H_m\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ $(m=1,2,\dots)$. Let, moreover, either

$$\limsup_{m\to+\infty}\sum_{l=1}^{\infty}\big(\|G_m(\tau_l)\|+\|u_m(\tau_l)\|\big)<+\infty\ \text{ or }\ \limsup_{m\to+\infty}\sum_{l=1}^{\infty}\|H_m(\tau_l+)-H_m(\tau_l)\|<+\infty.$$

Then inclusion (5) *holds.*

Corollary 2. *Let condition* (6) *hold and let the conditions*

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_m(\tau_l)\bigg)=B(t)-B(a),$$

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t H_m(\tau)P_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}(B(\tau_l+)-G_m(\tau_l+))\,G_m(\tau_l)\bigg)=\int\limits_a^t P_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}G_0(\tau_l),$$

$$\lim_{m\to+\infty}\bigg(\int\limits_a^t H_m(\tau)\,q_m(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}(B(\tau_l+)-G_m(\tau_l+))\,u_m(\tau_l)\bigg)=\int\limits_{t_0}^t q_0(\tau)\,d\tau+\sum_{\tau_l\in[a,t[}u_0(\tau_l)$$

hold uniformly on I, where $B\in\mathrm{BVAC}_{loc}(I,T;\mathbb{R}^{n\times n})$ and

$$H_m(t)\equiv I_n-\int\limits_a^t P_m(\tau)\,d\tau-\sum_{\tau_l\in[a,t[}G_m(\tau_l)+B(t)-B(a)\ \ (m=1,2,\dots).$$

Then inclusion (5) *holds.*

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On Well-Possedness of General Linear Boundary Value Problems for High Order Ordinary Linear Differential Equations

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We consider the question on the well-posedness of the boundary value problem

$$u^{(n)}=\sum_{l=1}^{n}p_l(t)u^{(l-1)}+p_0(t)\ \text{ for a.a. } t\in I, \tag{1}$$

$$\ell_i(u,u',\dots,u^{(n-1)})=c_{i0}\ \ (i=1,\dots,n), \tag{2}$$

where $I=[a,b]$ is an arbitrary closed interval from $\mathbb{R}$, $p_l\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $c_{io}\in\mathbb{R}$ $(i=1,\dots,n)$, and $\ell_i:\mathrm{AC}^{(n-1)}(I;\mathbb{R})\to\mathbb{R}$ $(i=1,\dots,n)$ are linear bounded functionals with respect to the norm

$$\|u\|_{\mathrm{AC}}=\sum_{j=1}^{n}\|u^{(j-1)}\|_c.$$

Here $\mathrm{AC}^{(n-1)}(I;\mathbb{R})$ is the set of all functions $u:I\to\mathbb{R}$ such that the derivatives $u^{(j)}$ $(j=0,\dots,n-1)$ are absolutely continuous functions on I, i.e., such that $u^{(j)}\in\mathrm{AC}(I;\mathbb{R})$ $(j=0,\dots,n-1)$, and $\|v\|_c=\max\{|v(t)|:t\in I\}$ for every continuous function $v:I\to\mathbb{R}$.

By $|||\ell|||$ we denote the usual norm of the linear operator ℓ.

Under a solution of the differential equation (1) we understand a function $u\in\mathrm{AC}^{(n-1)}(I;\mathbb{R})$ such that

$$u^{(n)}(t)=\sum_{l=1}^{n}p_l(t)u^{(l-1)}(t)+p_0(t)\ \text{ for a.a. } t\in I.$$

Let u_0 be the unique solution of the Cauchy problem $(1),(2)$.

Along with problem $(1),(2)$ consider the sequence of problems

$$u^{(n)}=\sum_{l=1}^{n}p_{lk}(t)u^{(l-1)}+p_{0k}(t)\ \text{ for a.a. } t\in I, \tag{1_k}$$

$$\ell_{ik}(u,u',\dots,u^{(n-1)})=c_{ik}\ \ (i=1,\dots,n), \tag{2_k}$$

$(k=1,2,\dots)$, where $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $c_{ik}\in\mathbb{R}$ $(i=1,\dots,n;\ k=1,2,\dots)$, and $\ell_{ik}:\mathrm{AC}^{(n-1)}(I;\mathbb{R})\to\mathbb{R}$ $(i=1,\dots,n;\ k=1,2,\dots)$ are linear bounded functionals.

Definition. We say that the sequence $(p_{1k},\dots,p_{nk},p_{0k};\ell_{1k},\dots,\ell_{nk})$ $(k=1,2,\dots)$ belongs to the set $\mathcal{S}(p_1,\dots,p_n,p_0;\ell_1,\dots,\ell_n)$ if for every $c_{i0}\in\mathbb{R}$ $(i=1,\dots,n)$ and a sequence $c_{ik}\in\mathbb{R}$ $(i=1,\dots,n;\ k=1,2,\dots)$, satisfying the condition

$$\lim_{k\to+\infty} c_{ik}=c_{i0}\ \ (i=1,\dots,n), \tag{3}$$

the boundary value problem $(1_k),(2_k)$ has the unique solution u_k for any natural k and

$$\lim_{k\to+\infty} u_k^{(i-1)}(t)=u_0^{(i-1)}(t)\ \ (i=1,\dots,n) \tag{4}$$

uniformly on I.

Along with equations (1) and (1_k) $(k=1,2,\dots)$ we consider the corresponding homogeneous equations

$$u^{(n)}=\sum_{l=1}^{n} p_l(t)u^{(i-1)}\ \text{ for a.a }\ t\in I \tag{1_0}$$

and

$$u^{(n)}=\sum_{l=1}^{n} p_{lk}(t)u^{(i-1)}\ \text{ for a.a }\ t\in I \tag{1_{0k}}$$

$(k=1,2,\dots)$.

If the functions $v_i\in \mathrm{AC}^{(n-1)}(I;\mathbb{R})$ $(i=1,\dots,n)$, then by

$$w_0(v_1,\dots,v_n)(t)=\det\big((v_i^{(l-1)}(t))_{i,l=1}^n\big)$$

we denote the so called Wronskiĭ's determinant, and by $w_{il}(v_1,\dots,v_n)(t)$ $(i,l=1,\dots,n)$ we denote the cofactor of the il-element of $w_0(v_1,\dots,v_n)$.

Let u_l $(l=1,\dots,n)$ and u_{lk} $(l=1,\dots,n;\ k=1,2,\dots)$ be the fundamental systems of solutions of the homogeneous systems (1_0) and (2_{0k}) $(k=1,2,\dots)$, respectively.

Below we give necessary and sufficient conditions, as well some sufficient conditions, guaranteeing the inclusion

$$\big((p_{1k},\dots,p_{nk},p_{0k};\ell_{1k},\dots,\ell_{nk})\big)_{k=1}^{+\infty}\in\mathcal{S}(p_1,\dots,p_n,p_0;\ell_1,\dots,\ell_n). \tag{5}$$

Theorem 1. *Let the functions $p_l\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and let the linear functionals ℓ_i, ℓ_{ik} $(i=1,\dots,n;\ k=1,2,\dots)$ be such that the conditions*

$$\lim_{k\to+\infty}\ell_{ik}(u,u',\dots,u^{(n-1)})=\ell_i(u,u',\dots,u^{(n-1)})\ \ \textit{for}\ \ u\in\mathrm{AC}^{(n-1)}(I;\mathbb{R})\ \ (i=1,\dots,n), \tag{6}$$

$$\limsup_{k\to+\infty}|||\ell_{ik}|||<+\infty\ \ (i=1,\dots,n) \tag{7}$$

hold. Then inclusion (5) *holds if and only if there exists a sequence of functions h_{il}, $h_{ilk}\in\mathrm{AC}(I;\mathbb{R})$ $(i,l=1,\dots,n;\ k=1,2,\dots)$ such that the conditions*

$$\inf\Big\{\big|\det((h_{il}(t))_{i,l=1}^n)\big|:\ t\in I\Big\}>0 \tag{8}$$

and

$$\limsup_{k\to+\infty}\sum_{i,l=1}^{n}\int_a^b\Big|h'_{ilk}(t)+h_{i\,l-1\,k}(t)\operatorname{sgn}(l-1)+h_{ink}(t)\,p_l(t)\Big|\,dt<+\infty \tag{9}$$

hold, and the conditions

$$\lim_{k\to+\infty} h_{ilk}(t)=h_{il}(t)\ \ (i,l=1,\dots,n) \tag{10}$$

and

$$\lim_{k\to+\infty}\int_a^t h_{ink}(\tau)p_{lk}(\tau)\,d\tau=\int_a^t h_{in}(\tau)p_l(\tau)\,d\tau\ \ (i=1,\dots,n;\ l=0,\dots,n)$$

hold uniformly on I.

Theorem 2. *Let the functions $p_l\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and let the linear functionals ℓ_i, ℓ_{ik} $(i=1,\dots,n;\ k=1,2,\dots)$ be such that conditions* (6) *and* (7) *hold. Then inclusion* (5) *holds if and only if the conditions*

$$\lim_{k\to+\infty} u_{lk}^{(i-1)}(t)=u_l^{(i-1)}(t)\ \ (i,l=1,\dots,n)$$

and

$$\lim_{k\to+\infty}\int_a^t \frac{w_{in}(u_{1k},\dots,u_{nk})(\tau)}{w_0(u_{1k},\dots,u_{nk})(\tau)}\,p_{0k}(\tau)\,d\tau=\int_a^t \frac{w_{in}(u_1,\dots,u_n)(\tau)}{w_0(u_1,\dots,u_n)(\tau)}\,p_0(\tau)\,d\tau\ \ (i=1,\dots,n) \tag{11}$$

hold uniformly on I.

Theorem 3. *Let the functions $p_l\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and let the linear functionals ℓ_i, ℓ_{ik} $(i=1,\dots,n;\ k=1,2,\dots)$ be such that conditions* (6), (7) *and*

$$\limsup_{k\to+\infty}\int_a^b \|p_{lk}(t)\|\,dt<+\infty\ \ (l=1,\dots,n)$$

hold, and the condition

$$\lim_{k\to+\infty}\int_a^t p_{lk}(\tau)\,d\tau=\int_a^t p_l(\tau)\,d\tau\ \ (l=0,\dots,n)$$

hold uniformly on I. Then the boundary value problem $(1_k),(2_k)$ has the unique solution u_k for any natural k and condition (4) *holds uniformly on I.*

Corollary 1. *Let the functions $p_l\in L(I;\mathbb{R})$ $(l=0,\dots,n)$, $p_{lk}\in L(I;\mathbb{R})$ $(l=0,\dots,n;\ k=1,2,\dots)$ and let the linear functionals ℓ_i, ℓ_{ik} $(i=1,\dots,n;\ k=1,2,\dots)$ be such that conditions* (3), (6), (7) *and* (9) *hold, and conditions* (10) *and*

$$\lim_{k\to+\infty}\int_a^t h_{ink}(\tau)p_{lk}(\tau)\,d\tau=\int_a^t p_l^*(\tau)\,d\tau\ \ (i=1,\dots,n;\ l=0,\dots,n)$$

hold uniformly on I, where $p_l^*\in L(I;\mathbb{R})$ $(l=0,\dots,n)$; h_{il}, $h_{ilk}\in \mathrm{AC}(I;\mathbb{R})$ $(i,l=1,\dots,n;$ $k=1,2,\dots)$. Then the inclusion

$$\big((p_{1k},\dots,p_{nk},p_{0k};\ell_{1k},\dots,\ell_{nk})\big)_{k=1}^{+\infty}\in\mathcal{S}(p_1-p_1^*,\dots,p_n-p_n^*,p_0-p_0^*;\ell_1,\dots,\ell_n)$$

holds.

Remark. In Theorem 2 and Corollary 1, without loss of generality we can assume that $h_{ii}(t)\equiv 1$ and $h_{il}(t)\equiv 0$ $(i\neq l;\ i,l=1,\dots,n)$. So condition (8) is valid evidently.

Remark. If $n=2$ in Theorem 3, then condition (11) has the form

$$\lim_{k\to+\infty}\int_a^t \frac{u_{1k}'(\tau)p_{0k}(\tau)}{u_{1k}(\tau)u_{2k}'(\tau)-u_{2k}(\tau)u_{1k}'(\tau)}\,d\tau=\int_a^t \frac{u_1'(\tau)p_0(\tau)}{u_1(\tau)u_2'(\tau)-u_2(\tau)u_1'(\tau)}\,d\tau,$$
$$\lim_{k\to+\infty}\int_a^t \frac{u_{1k}(\tau)p_{0k}(\tau)}{u_{1k}(\tau)u_{2k}'(\tau)-u_{2k}(\tau)u_{1k}'(\tau)}\,d\tau=\int_a^t \frac{u_1(\tau)p_0(\tau)}{u_1(\tau)u_2'(\tau)-u_2(\tau)u_1'(\tau)}\,d\tau.$$

In the equalities we can take u_{2k} instead of u_{1k} $(k=1,2,\dots)$ and u_2 instead of u_1.

For the proof we use the well-know concept. It is well known that if the function u is a solution of problem (1), (2), then the vector-function $x=(x_i)_{i=1}^n$, $x_i=u^{(i-1)}$ $(i=1,\dots,n)$ is a solution of the following general linear boundary value problem for system of ordinary differential equations

$$\frac{dx}{dt}=P(t)\,x+q(t),$$
$$\ell(x)=c_0,$$

where the matrix- and vector-functions $P(t)=(p_{il}(t))_{i,l=1}^n$ and $q(t)=(q_i(t))_{i=1}^n$ are defined, respectively, by

$$p_{il}(t)\equiv 0,\ \ p_{i\,i+1}\equiv 1\ \ (l\neq i+1;\ i=1,\dots,n-1;\ l=1,\dots,n),$$
$$p_{nl}(t)\equiv p_l(t)\ \ (l=1,\dots,n);$$
$$q_i(t)\equiv 0\ \ (i=1,\dots,n-1),\ \ q_n(t)\equiv p_0(t);$$
$$\ell(x)=\big(\ell_l(u,u',\dots,u^{(n-1)})\big)_{l=1}^n\ \ (x=(u^{(l-1)})_{l=1}^n);\ \ c_0=(c_{l0})_{l=1}^n.$$

Analogously, problem (1_k), (2_k) can be rewritten in the form of the last type problem for every natural k. So, using the results contained in [1–3] we get the results given above.

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A Multi-Point Problem for Second Order Differential Equation with Piecewise-Constant Argument of Generalized Type

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In $[0,T]$ we consider the multi-point problem for the second order differential equation with piecewise-constant argument of generalized type

$$\ddot{x}=a_1(t)\dot{x}(t)+a_2(t)x(t)+a_3(t)\dot{x}(\gamma(t))+a_4(t)x(\gamma(t))+f(t), \tag{1}$$

$$\sum_{j=0}^{N}\big\{b_{1j}\dot{x}(\theta_j)+c_{1j}x(\theta_j)\big\}=d_1, \tag{2}$$

$$\sum_{j=0}^{N}\big\{b_{2j}\dot{x}(\theta_j)+c_{2j}x(\theta_j)\big\}=d_2, \tag{3}$$

where $x(t)$ is unknown function, the functions $a_i(t)$, $i=\overline{1,4}$ and $f(t)$ are continuous on $[0,T]$; $0=\theta_0<\theta_1<\cdots<\theta_{N-1}<\theta_N=T$, $\theta_j\le\zeta_j\le\theta_{j+1}$ for all $j=0,1,\cdots,N-1$: $\gamma(t)=\zeta_j$ if $t\in[\theta_j,\theta_{j+1})$, $j=\overline{0,N-1}$; b_{sj}, c_{sj} and d_s are constants, where $s=1,2$; $j=\overline{0,N}$.

A solution to problem (1)–(3) is a function $x(t)$, twice continuously differentiable on $[0,T]$, it satisfies equation (1) and the multi-point conditions (2), (3).

The study of differential equations with piecewise-constant argument began with the works by Cook, Busenberg, Wiener, and Shah [11–13,27,28]. Many researchers have extensively studied the questions of the existence and uniqueness of solutions, oscillations and stability, integral manifolds and periodic solutions, etc. Differential equations with piecewise-constant argument have been used to develop various models in biology, mechanics, and electronics.

When models are described by differential equations with piecewise-constant argument, the deviation of the argument values is always constant and equal to one, since the greatest integer function is taken as the deviation of the argument. But this approach can contradict real phenomena. In the works by Akhmet [2–4], the greatest integer function as deviating argument was replaced by an arbitrary piecewise constant function. Thus, differential equations with piecewise-constant argument of generalized type are more suitable for modeling and solving various application problems, including areas of neural networks, discontinuous dynamical systems, hybrid systems, etc. To date, the theory of differential equations with piecewise-constant argument of generalized type on the entire axis has been developed and their applications have been implemented. The results were extended to periodic impulse systems of differential equations with piecewise-constant argument of generalized type [5–10]. Along with the study of various properties of differential equations with piecewise-constant argument, a number of authors investigated the questions of solvability and construction of solutions to boundary value problems for these equations on a finite interval [14,19–23,25,26,29–31]. Particular attention was paid to periodic and multi-point problems for second order differential equations with piecewise-constant argument due to their wide application to natural sciences and engineering [1,18,24].

Although the theory of boundary value problems for differential equations with piecewise-constant argument has been developed by a number of researchers, the question of solvability of boundary value problems for systems of differential equations with piecewise-constant argument of generalized type on a finite interval still remains open.

Therefore, the questions of solvability of boundary value problems for such equations are of great importance and relevance. The construction of new general solutions to second order differential equations with piecewise-constant argument of generalized type and investigation into their properties provides an opportunity to solve new classes of problems.

In the present paper, the ideas and results of [15–17] are extended to second order differential equations with piecewise-constant argument of generalized type. We study conditions for unique solvability of multi-point problem for second order differential equation with piecewise-constant argument of generalized type (1)–(3) and construct the algorithms for finding its solution. For this we use the Dzhumabaev parameterization method [15]. The results can be used in the numerical solving of application problems [16].

At first, we introduce new functions $z^{(1)}(t)=x(t)$, $z^{(2)}(t)=\dot{x}(t)$ and rewrite problem (1)–(3) in the following form

$$\dot{z}=A(t)z(t)+A_0(t)z(\gamma(t))+g(t), \tag{4}$$

$$\sum_{j=0}^{N} C_j z(\theta_j)=d, \tag{5}$$

where $z(t)=col(z^{(1)}(t),z^{(2)}(t))$ is unknown vector function,

$$A(t)=\begin{pmatrix} 0 & 1\\ a_2(t) & a_1(t)\end{pmatrix},\quad A_0(t)=\begin{pmatrix} 0 & 0\\ a_4(t) & a_3(t)\end{pmatrix},\quad g(t)=\begin{pmatrix} 0\\ f(t)\end{pmatrix},$$

$$C_j=\begin{pmatrix} c_{1j} & b_{1j}\\ c_{2j} & b_{2j}\end{pmatrix},\ \ j=\overline{0,N},\quad d=\begin{pmatrix} d_1\\ d_2\end{pmatrix}.$$

A solution to problem (4), (5) is a two-dimensional vector function $z(t)$ which is continuously differentiable on $[0,T]$, it satisfies system (4) and the multi-point condition (5).

Denote by Δ_N a partition of the interval $[0,T)$: $[0,T)=\bigcup\limits_{r=1}^{N}[\theta_{r-1},\theta_r)$ by lines $t=\theta_j$, $j=\overline{1,N-1}$. Let $z_r(t)$ be a restriction of function $z(t)$ on rth interval $[\theta_{r-1},\theta_r)$, i.e. $z_r(t)=z(t)$ for $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,N}$. Then problem (4), (5) reduce to the following equivalent problem

$$\dot{z}_r=A(t)z_r(t)+A_0(t)z_r(\zeta_{r-1})+g(t),\ \ t\in[\theta_{r-1},\theta_r),\ \ r=\overline{1,N}, \tag{6}$$

$$\sum_{j=0}^{N-1} C_j z_{j+1}(\theta_j)+C_N\lim_{t\to T-0} z_N(t)=d, \tag{7}$$

$$\lim_{t\to\theta_p-0} z_p(t)=z_{p+1}(\theta_p),\ \ p=\overline{1,N-1}. \tag{8}$$

In (4) we take into account that $\gamma(t)=\zeta_j$ if $t\in[\theta_j,\theta_{j+1})$, $j=\overline{0,N-1}$. Condition (8) is the continuity condition of function $z(t)$ on the interior lines $t=t_p$, $p=0,1,2,\dots,N-1$.

Introduce additional parameters $\lambda_r=z_r(\zeta_{r-1})$ for all $r=\overline{1,N}$. On every rth interval we change function $z_r(t)$ by $u_r(t)=z_r(t)-\lambda_r \quad r=\overline{1,N}$.

Then, from (6)–(8), we obtain the following problem with parameters

$$\dot{u}_r=A(t)u_r(t)+[A(t)+A_0(t)]\lambda_r+g(t),\ \ t\in[\theta_{r-1},\theta_r),\ \ u_r(\zeta_{r-1})=0,\ \ r=\overline{1,N}, \tag{9}$$

$$\sum_{j=0}^{N-1} C_j\lambda_{j+1}+\sum_{j=0}^{N-1} C_j u_{j+1}(\theta_j)+C_N\lambda_N+C_N\lim_{t\to T-0} u_N(t)=d, \tag{10}$$

$$\lambda_p+\lim_{t\to\theta_p-0} z_p(t)=\lambda_{p+1}+z_{p+1}(\theta_p),\ \ p=\overline{1,N-1}. \tag{11}$$

Problems (9) are the Cauchy problems for a system of ordinary differential equations with parameters. Conditions (10), (11) are the relations for determining unknown parameters λ_r, $r=\overline{1,N}$.

Let $X_r(t)$ be a fundamental matrix of differential equation $\dot{u}_r = A(t)u_r(t)$ for $t\in[\theta_{r-1},\theta_r)$, $r=\overline{1,N}$. Then, solutions of the Cauchy problems (9) have the following form

$$u_r(t)=X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)[A(\tau)+A_0(\tau)]\,d\tau\lambda_r \\ + X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)g(\tau)\,d\tau,\ \ t\in[\theta_{r-1},\theta_r),\ \ r=\overline{1,N}. \tag{12}$$

Substituting right-hand side of (12) for $t=\theta_j$, $j=\overline{0,N-1}$, $t=T$ to (10), (11), we have

$$\sum_{j=0}^{N-1} C_j[I+D_{j+1}(\theta_j)]\lambda_{j+1}+C_N[I+D_N(T)]\lambda_N=d-\sum_{j=0}^{N-1}C_jF_{j+1}(\theta_j)-C_NF_N(T), \tag{13}$$

$$[I+D_p(\theta_p)]\lambda_p-[I+D_{p+1}(\theta_p)]\lambda_{p+1}=F_{p+1}(\theta_p)-F_p(\theta_p),\ \ p=\overline{1,N-1}, \tag{14}$$

where I is a unit matrix,

$$D_r(t)=X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)[A(\tau)+A_0(\tau)]\,d\tau,$$

$$F_r(t)=X_r(t)\int\limits_{\zeta_{r-1}}^{t} X_r^{-1}(\tau)g(\tau)\,d\tau,\ \ t\in[\theta_{r-1},\theta_r),\ \ r=\overline{1,N}.$$

We rewrite equations (13), (14) in the following form

$$Q(\Delta_N)\lambda=F(\Delta_N),\ \ \lambda\in R^{2N}. \tag{15}$$

Definition 1. Problem (1)–(3) is called uniquely solvable if, for any triple $(f(t),d_1,d_2)$, where $f(t)\in C([0,T],R)$ and $d_1,d_2\in R$, it has a unique solution.

Theorem 1. *Problem* (1)–(3) *is solvable if and only if the vector* $F(\Delta_N)$ *is orthogonal to the kernel of the transposed matrix* $(Q(\Delta_N))'$, *i.e., for any* $\xi\in Ker(Q(\Delta_N))'$, *the following equality is true:* $(F(\Delta_N),\xi)=0$, *where* $(\cdot,\cdot)$ *is the scalar product in* R^{2N}.

Theorem 2. *Problem* (1)–(3) *is uniquely solvable if and only if the* $(2N\times 2N)$ *matrix* $Q(\Delta_N)$ *is invertible.*

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Existence and Uniqueness Theorems to Generalized Emden–Fowler Type Equations

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Abstract. For generalized Emden–Fowler type equations we obtain conditions on initial values providing uniqueness or non-uniqueness of solutions.

1 Introduction and Basic Notation

Consider the equation

$$y'' = p(x, y, y')|y|_{\pm}^{k_0}|y'|_{\pm}^{k_1}, \tag{1.1}$$

where $|a|_{\pm}^{b}$ denotes $|a|^b \operatorname{sgn} a$ and a positive continuous function p is locally Lipschitz continuous in the last two arguments. The real constants k_0 and k_1 are positive.

Given any $x_0, y_0, y_1 \in \mathbf{R}$, equation (1.1) has a solution defined in a neighborhood of $x_0 \in \mathbf{R}$ and satisfying the initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1. \tag{1.2}$$

Our purpose is to know whether or not the above solution is unique. To obtain results, we use some methods of [1]. In some simple cases the results coincide with those of [2] and [3].

Without loss of generality, suppose $x_0 = 0$. Put

$$\begin{gathered}
p_0 = p(0,0,0) > 0, \\
p_m(X) = \inf\big\{p(x,u,v): \ |x| \le X, \ |u| \le X, \ |v| \le X\big\}, \\
p_M(X) = \sup\big\{p(x,u,v): \ |x| \le X, \ |u| \le X, \ |v| \le X\big\},
\end{gathered}$$

and note that $p_m(X) \to p_0$ and $p_M(X) \to p_0$ as $X \to +0$.

Since p is locally Lipschitz continuous in the last two arguments, we may assume it to satisfy the inequalities

$$|p(x,u,v) - p(x,w,v)| \le p_0\lambda_X|u-w| \ \text{ and } \ |p(x,u,v) - p(x,u,w)| \le p_0\lambda_X|v-w|$$

for some $\lambda_X > 0$ and for all real $x, u, v, w \in [-X; X]$.

2 Main Results

Theorem 2.1. *If $k_0 \in (0;1)$, $y_0 = 0$, $y_1 \ne 0$, then in a neighborhood of* 0 *equation* (1.1) *has a unique solution satisfying* (1.2).

Theorem 2.2. *If $k_1 \in (0;1)$, $y_0 \neq 0$, $y_1 = 0$, then equation* (1.1) *has at least two solutions satisfying* (1.2) *and differing at points arbitrarily close to* 0.

Theorem 2.3. *If $k_0 > 0$, $k_1 > 0$, $k_0 + k_1 \geq 1$, $y_0 = y_1 = 0$, then in a neighborhood of* 0 *equation* (1.1) *has a unique solution satisfying* (1.2).

Theorem 2.4. *If $k_0, k_1, k_0 + k_1 \in (0;1)$ and $y_0 = y_1 = 0$, then in a neighborhood of* 0 *equation* (1.1) *has at least two solutions satisfying* (1.2) *and differing at points arbitrarily close to* 0.

3 Proofs

Proof of Theorem 2.1. According to the equation and initial conditions, we have

$$y(x) = \int_0^x y'(\xi)\,d\xi \ \text{ and } \ y'(x) = y_1 + \int_0^x p\bigg(\eta, \int_0^\eta y'(\xi)\,d\xi,\, y'(\eta)\bigg) \bigg| \int_0^\eta y'(\xi)\,d\xi \bigg|_\pm^{k_0} |y'(\eta)|_\pm^{k_1}\,d\eta.$$

The last expression can be written as $F(y', y', y', y')(x)$, where

$$F(u_1, u_2, u_3, u_4)(x) = y_1 + \int_0^x p\bigg(\eta, \int_0^\eta u_1(\xi)\,d\xi, u_2(\eta)\bigg) \bigg| \int_0^\eta u_3(\xi)\,d\xi \bigg|_\pm^{k_0} |u_4(\eta)|_\pm^{k_1}\,d\eta$$

for any continuous functions u_1, u_2, u_3, u_4.

Suppose y and z are different solutions to (1.1), (1.2). There exists a segment $[-X;X]$ with $0 < X < 1$ such that both $y'(x)/y_1$ and $z'(x)/y_1$ are contained in $[\frac{1}{2}\,;2]$ for any $x \in [-X;X]$.

Put $\delta = \sup\{|y'(x) - z'(x)| :\ x \in [-X;X]\}$. We have

$$\begin{aligned}
|y'(x) - z'(x)| &= \big|F(y', y', y', y')(x) - F(z', z', z', z')(x)\big| \\
&\leq \big|F(y', y', y', y')(x) - F(y', y', y', z')(x)\big| + \big|F(y', y', y', z')(x) - F(y', y', z', z')(x)\big| \\
&\quad + \big|F(y', y', z', z')(x) - F(y', z', z', z')(x)\big| + \big|F(y', z', z', z')(x) - F(z', z', z', z')(x)\big|.
\end{aligned}$$

Now we estimate, on $[-X;X]$, each summand of the last sum. For the second one, we use the inequality

$$\big||a|_\pm^k - |b|_\pm^k\big| \leq \frac{k|a-b|}{\min\{|a|, |b|\}^{1-k}} \ \text{ whenever } \ 0 < k < 1 \ \text{ and } \ \operatorname{sgn} a = \operatorname{sgn} b \neq 0.$$

So,

$$\begin{aligned}
\big|F(y', y', y', y')(x) - F(y', y', y', z')(x)\big| &\leq X \cdot p_M(X) \cdot |2y_1X|^{k_0} \cdot k_1|y_1|^{k_1-1} 2^{|k_1-1|}\delta, \\
\big|F(y', y', y', z')(x) - F(y', y', z', z')(x)\big| &\leq p_M(X) \cdot \frac{k_0}{k_0+1}\, X^{k_0+1} \Big|\frac{2}{y_1}\Big|^{1-k_0} \delta \cdot |2y_1|^{k_1}, \\
\big|F(y', y', z', z')(x) - F(y', z', z', z')(x)\big| &\leq X \cdot p_0\lambda_X\delta \cdot |2y_1X|^{k_0} \cdot |2y_1|^{k_1}, \\
\big|F(y', z', z', z')(x) - F(z', z', z', z')(x)\big| &\leq X \cdot Xp_0\lambda_X\delta \cdot |2y_1X|^{k_0} \cdot |2y_1|^{k_1}.
\end{aligned}$$

Now we choose $X > 0$ small enough to make each right-hand side of the four inequalities less than $\delta/8$. This yields $|y'(x) - z'(x)| < \delta/2$ on $[-X;X]$, contradicting to the definition of δ. $\square$

Proof of Theorem 2.2. Without loss of generality we assume $y_0 > 0$.

The first solution to (1.1), (1.2) is evident: $y \equiv y_0$. To find another one, put $\alpha = \frac{1}{1-k_1} > 1$ and consider the first-order 2-dimensional system

$$\begin{cases} y'(x) = |v(x)|^\alpha, \\ v'(x) = \dfrac{|y(x)|_\pm^{k_0}}{\alpha}\, p\big(x, y(x), |v(x)|^\alpha\big) \end{cases}$$

with the initial conditions $y(0) = y_0$, $v(0) = 0$.

Since $y_0 \neq 0$, this initial value problem is regular in a neighborhood of the point $(0, y_0, 0)$ regardless of whether or not k_0 is less than 1. Hence the problem has a solution defined in a neighborhood of 0. It follows from the second equation of the system that $v'(0) \neq 0$ and therefore $y'(x)$, which equals $|v(x)|^\alpha$, vanishes at 0 but cannot be identically zero in any neighborhood of 0. So, y cannot be constant.

Further, $y(x)$, $v(x)$, and $y'(x)$ are positive for $x > 0$ and

$$y''(x) = \alpha v(x)^{\alpha-1}\, \frac{y(x)^{k_0}}{\alpha} p(x, y(x), v(x)^\alpha) = y'(x)^{(\alpha-1)/\alpha}\, y(x)^{k_0} p(x, y(x), y'(x)).$$

Since $(\alpha-1)/\alpha = k_1$, the function $y(x)$ is a solution to (1.1), (1.2) other than the constant one. $\square$

Proof of Theorem 2.3. The existence of a solution is evident even without the Peano existence theorem since $y \equiv 0$ surely satisfies both (1.1) and (1.2). So, we have to prove that no other solution exists in a sufficiently small neighborhood of 0.

First, consider constant-sign solutions to (1.1), (1.2) with constant-sign derivative in a half-neighborhood of 0. Here we have the following equivalences for such solutions (as $x \to 0$):

$$y''(x)|y'(x)|^{1-k_1} \sim p_0\, |y(x)|_\pm^{k_0}\, y'(x),$$

$$\begin{cases} \big(\log|y'| \operatorname{sgn} y'\big)'(x) \sim \dfrac{p_0}{k_0+1}\, \big(|y|^{k_0+1}\big)'(x) & \text{if } k_1 = 2, \\ \big(|y'|_\pm^{2-k_1}\big)'(x) \sim \dfrac{(2-k_1)p_0}{k_0+1}\, \big(|y|^{k_0+1}\big)'(x) & \text{if } k_1 \neq 2. \end{cases}$$

The right-hand sides of the two last equivalences are the derivatives of bounded functions. The same must be true for equivalent functions. But in the case $k_1 \geq 2$, the left-hand sides are the derivatives of unbounded functions. Because of this contradiction, we go on with the case $k_1 < 2$ only. By L'Hôpital's rule, the last equivalence invokes

$$|y'(x)|_\pm^{2-k_1} \sim \frac{(2-k_1)p_0}{k_0+1}\, |y(x)|^{k_0+1},$$

$$y'(x) \sim \Big(\frac{(2-k_1)p_0}{k_0+1}\Big)^{1/(2-k_1)} |y(x)|^{(k_0+1)/(2-k_1)},$$

$$\begin{cases} \big(\log|y|\, \operatorname{sgn} y\big)'(x) \sim \Big(\dfrac{(2-k_1)p_0}{k_0+1}\Big)^{1/(2-k_1)} & \text{if } k_0+1 = 2-k_1, \\ \big(|y|_\pm^{1-(k_0+1)/(2-k_1)}\big)'(x) \sim \Big(\dfrac{(2-k_1)p_0}{k_0+1}\Big)^{1/(2-k_1)} \Big(1 - \dfrac{k_0+1}{2-k_1}\Big) & \text{if } k_0+1 \neq 2-k_1. \end{cases}$$

By the conditions of the theorem, the exponent of $|y|_\pm$ in the last equivalence, which equals

$$1 - \frac{k_0+1}{2-k_1} = \frac{1-k_0-k_1}{2-k_1}\,,$$

is negative. Hence, the left-hand sides of the last two equivalences are the derivatives of unbounded functions but are equivalent to finite constants. This contradiction shows that in any half-neighborhood of 0 there is no constant-sign solution to (1.1), (1.2) with constant-sign derivative, besides the trivial solution $y \equiv 0$.

Now, what about non-constant-sign solutions? If such a solution pretends to disprove the statement of the theorem, its domain must include a monotonic sequence of disjoint intervals $(a_j; b_j)$ such that

(i) $y(x)y'(x) \neq 0$ on $(a_j; b_j)$,

(ii) $y(a_j)y'(a_j) = 0$,

(iii) $y(b_j)y'(b_j) = 0$,

(iv) $a_j \to 0$ and $b_j \to 0$ as $j \to \infty$.

Note that neither $y(a_j) = y'(a_j) = 0$ nor $y(b_j) = y'(b_j) = 0$ can hold because of the first part of our proof. Neither $y(a_j) = y(b_j) = 0$ nor $y'(a_j) = y'(b_j) = 0$ can hold because of condition (i), Rolle's lemma, and equation (1.1). If $y(a_j) = 0$ and $y'(a_j) > 0$, then, according to (1.1), we have $y(x) > 0$, $y'(x) > 0$, and $y''(x) > 0$ on $(a_j; b_j)$, which makes (iii) impossible. Similarly, if $y(b_j) = 0$ and $y'(b_j) > 0$, then we have $y(x) < 0$, $y'(x) > 0$, and $y''(x) < 0$ on $(a_j; b_j)$, which also makes (iii) impossible. So, only the cases $y(a_j) = 0$, $y'(a_j) < 0$, $y(b_j) < 0$, $y'(b_j) = 0$ and $y(a_j) > 0$, $y'(a_j) = 0$, $y(b_j) = 0$, $y'(b_j) < 0$ are possible. A pair of such segments can match at a common end-point with $y(x) = 0$. But outside their union the solution can only stay constant or move away from zero. Thus, it cannot satisfy (1.2). □

Proof of Theorem 2.4. The first solution to (1.1), (1.2) is $y \equiv 0$. To find another one, put $\beta = \frac{k_0+1}{1-k_0-k_1} > 1$ and consider the operators acting on the space of positive continuous functions by the following formulae with $u \in C[0; X]$, $X > 0$, and $x \in [0; X]$:

$$\begin{aligned}
Y(u)(x) &= \int_0^x s^\beta u(s)\,ds,\\
P(u)(x) &= p\big(x, Y(u)(x), x^\beta u(x)\big),\\
Q(u)(x) &= Y(u)(x)^{k_0} \cdot (x^\beta u(x))^{k_1} \cdot P(u)(x),\\
F(u)(x) &= x^{-\beta} \int_0^x Q(u)(s)\,ds.
\end{aligned}$$

The last one can be well defined also for $x = 0$ and can be shown to be a contraction. Thus, F has a unique fixed point, i.e. a positive continuous function u on $[0; X]$ such that $F(u) = u$.

Consider the function $y = Y(u)$. According to the definition of the operator Y, we have $y(0) = y'(0) = 0$. Further,

$$y'(x) = x^\beta u(x) = x^\beta F(u)(x) = \int_0^x Q(u)(s)\,ds,$$

whence

$$\begin{aligned}
y''(x) = Q(y)(x) &= Y(u)(x)^{k_0} \cdot (x^\beta u(x))^{k_1} \cdot P(u)(x)\\
&= y(x)^{k_0} y'(x)^{k_1} p\big(x, Y(u)(x), x^\beta u(x)\big) = y(x)^{k_0} y'(x)^{k_1} p(x, y(x), y'(x)).
\end{aligned}$$

So, y is a solution to (1.1), (1.2). It is positive on $(0; X]$ and therefore is just another solution from the statement of the theorem. □

4 Summary

$n = 2$	$0,\, 0$	$Y_0,\, 0$	$0,\, Y_1$	$Y_0,\, Y_1$
$k_0 \geq 1,\, k_1 \geq 1$	U	U	U	U
$k_0 < 1,\, k_1 \geq 1$	U: Th2.3	U	U: Th2.1	U
$k_0 \geq 1,\, k_1 < 1$	U: Th2.3	N: Th2.2	U	U
$k_0 + k_1 \geq 1,\, k_0 < 1,\, k_1 < 1$	U: Th2.3	N: Th2.2	U: Th2.1	U
$k_0 + k_1 < 1$	N: Th2.4	N: Th2.2	U: Th2.1	U

The first column of the above table contains conditions on the positive coefficients k_j. The first row describes initial data, $y(0)$ and $y'(0)$, with Y_0 and Y_1 denoting any non-zero value. In the main part of the table, "U" denotes the uniqueness of solutions to (1.1), (1.2) under the related conditions. "N" denotes non-uniqueness. These labels are followed by references to the related theorems. If not, then the classical existence and uniqueness theorem is implied.

Remark. Asymptotic behavior of unbounded solutions to equation (1.1) with additional conditions

$$0 < p_* \leq p(x,u,v) \leq p^* < \infty, \ \text{ for some } \ p_*, p^* \in \mathbb{R} \ \text{ and all } \ (x,u,v) \in \mathbb{R}^3,$$

is obtained in [4]. Asymptotic behavior of the first derivatives of bounded solutions is described in [5].

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On Generic Inhomogeneous Boundary-Value Problems for Differential Systems in Sobolev spaces

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Let a finite interval $[a,b]\subset\mathbb{R}$ and parameters $\{m,n,r,l\}\subset\mathbb{N}$, $1\le p\le\infty$, be given. By $W_p^{n+r}=W_p^{n+r}([a,b];\mathbb{C}):=\{y\in C^{n+r-1}[a,b]:\ y^{(n+r-1)}\in AC[a,b],\ y^{(n+r)}\in L_p[a,b]\}$ we denote a complex Sobolev space and set $W_p^0:=L_p$. This space is a Banach one with respect to the norm

$$\|y\|_{n+r,p}=\sum_{k=0}^{n+r-1}\|y^{(k)}\|_p+\|y^{(n+r)}\|_p,$$

where $\|\cdot\|_p$ is the norm in space $L_p([a,b];\mathbb{C})$. Similarly, by $(W_p^{n+r})^m:=W_p^{n+r}([a,b];\mathbb{C}^m)$ and $(W_p^{n+r})^{m\times m}:=W_p^{n+r}([a,b];\mathbb{C}^{m\times m})$ we denote Sobolev spaces of vector-valued functions and matrix-valued functions, respectively, whose elements belong to the function space W_p^{n+r}.

We consider the following linear boundary-value problem

$$(Ly)(t):=y^{(r)}(t)+\sum_{j=1}^{r}A_{r-j}(t)y^{(r-j)}(t)=f(t),\ \ t\in(a,b), \tag{1}$$

$$By=c, \tag{2}$$

where the matrix-valued functions $A_{r-j}(\cdot)\in(W_p^n)^{m\times m}$, the vector-valued function $f(\cdot)\in(W_p^n)^m$, the vector $c\in\mathbb{C}^l$, the linear continuous operator

$$B:(W_p^{n+r})^m\to\mathbb{C}^l \tag{3}$$

are arbitrarily chosen; and the vector-valued function $y(\cdot)\in(W_p^{n+r})^m$ is unknown.

We represent vectors and vector-valued functions in the form of columns. A solution to the boundary-value problem $(1),(2)$ is understood as a vector-valued function $y(\cdot)\in(W_p^{n+r})^m$ satisfying equation (1) almost everywhere on (a,b) (everywhere for $n\ge 2$) and equality (2) specifying l scalar boundary conditions. The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the boundary condition (2) with continuous operator (3) is the most general condition for this equation.

It includes all known types of classical boundary conditions, namely, the Cauchy problem, two- and multi-point problems, integral and mixed problems, and numerous nonclassical problems. The last class of problems may contain derivatives (generally fractional) $y^{(k)}(\cdot)$ with $0<k\le n+r$.

For $1\le p<\infty$, every operator B in (3) admits a unique analytic representation

$$By=\sum_{k=0}^{n+r-1}\alpha_k y^{(k)}(a)+\int_a^b\Phi(t)y^{(n+r)}(t)\,dt,\ \ y(\cdot)\in(W_p^{n+r})^m,$$

where the matrices $\alpha_k\in\mathbb{C}^{rm\times m}$ and the matrix-valued function $\Phi(\cdot)\in L_{p'}([a,b];\mathbb{C}^{rm\times m})$, $1/p+1/p'=1$.

For $p=\infty$ this formula also defines an operator $B:(W_\infty^{n+r})^m\to\mathbb{C}^{rm}$. However, there exist other operators from this class generated by the integrals over finitely additive measures.

With the generic inhomogeneous boundary-value problem (1), (2), we associate a linear continuous operator in pair of Banach spaces

$$(L,B):(W_p^{n+r})^m\to(W_p^n)^m\times\mathbb{C}^l. \tag{4}$$

Recall that a linear continuous operator $T\colon X\to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If operator T is Fredholm, then its range $T(X)$ is closed in Y and the index

$$\operatorname{ind}T:=\dim\ker T-\dim(Y/T(X))\in\mathbb{Z}$$

is finite.

Theorem 1. *The linear operator* (4) *is a bounded Fredholm operator with index $mr-l$.*

Theorem 1 allows the next specification.

For each number $k\in\{1,\dots,r\}$, we consider the family of the matrix Cauchy problems:

$$Y_k^{(r)}(t)+\sum_{j=1}^{r}A_{r-j}(t)Y_k^{(r-j)}(t)=O_m,\ \ t\in(a,b),$$

with the initial conditions

$$Y_k^{(j-1)}(a)=\delta_{k,j}I_m,\ \ j\in\{1,\dots,r\}.$$

Here, $Y_k(\,\cdot\,)$ is an unknown $m\times m$ matrix-valued function, and $\delta_{k,j}$ is the Kronecker symbol.

By $[BY_k]$ we denote the numerical $m\times l$ matrix, in which j-th column is the result of action of the operator B on the j-th column of the matrix-valued function $Y_k(\,\cdot\,)$.

Definition 1. A block rectangular numerical matrix $M(L,B):=([BY_0],\dots,[BY_{r-1}])\in\mathbb{C}^{mr\times l}$ is characteristic to the inhomogeneous boundary-value problem (1), (2). It consists of r rectangular block columns $[BY_k(\,\cdot\,)]\in\mathbb{C}^{m\times l}$.

Here mr is the number of scalar differential equations of system (1), and l is the number of scalar boundary conditions.

Theorem 2. *The dimensions of the kernel and cokernel of operator* (4) *are equal to the dimensions of the kernel and cokernel of the characteristic matrix $M(L,B)$, respectively.*

Theorem 2 implies a criterion for the invertibility of the operator (4).

Corollary 1. *The operator (L,B) is invertible if and only if $l=mr$ and the matrix $M(L,B)$ is nondegenerate.*

With problem (1), (2), we consider the sequence of boundary-value problems

$$L(k)y(t,k):=y^{(r)}(t,k)+\sum_{j=1}^{r}A_{r-j}(t,k)y^{(r-j)}(t,k)=f(t,k),\ \ t\in(a,b), \tag{5}$$

$$B(k)y(\,\cdot\,,k)=c(k),\ \ k\in\mathbb{N}, \tag{6}$$

where the matrix-valued functions $A_{r-j}(\,\cdot\,,k)$, the vector-valued function $f(\,\cdot\,,k)$, the vector $c(k)$, and a linear continuous operator $B(k)$ satisfy the above conditions to problem (1), (2).

With the boundary-value problem (5), (6), we associate a sequence of linear continuous operators

$$(L(k),B(k)):(W_p^{n+r})^m\to(W_p^n)^m\times\mathbb{C}^l$$

and a sequence of characteristic matrices depending on the parameter $k\in\mathbb{N}$

$$M(L(k),B(k)):=\Big(\big[B(k)Y_0(\,\cdot\,,(k))\big],\dots,\big[B(k)Y_{r-1}(\,\cdot\,,(k))\big]\Big)\subset\mathbb{C}^{mr\times l}.$$

We now formulate a sufficient condition for the convergence of the characteristic matrices $M(L(k),B(k))$ to the matrix $M(L,B)$.

Theorem 3. *If the sequence of operators $(L(k),B(k))$ converges strongly to the operator (L,B) for $k\to\infty$, then the sequence of characteristic matrices $M(L(k),B(k))$ converges to the matrix $M(L,B)$.*

Theorem 3 implies the next result.

Corollary 2. *Under the assumptions from Theorem 3, the following inequalities hold for sufficiently large k:*

$$\begin{gathered}\dim\ker(L(k),B(k))\le\dim\ker(L,B),\\ \dim\operatorname{coker}(L(k),B(k))\le\dim\operatorname{coker}(L,B).\end{gathered}$$

In particular:

1. If $l=mr$ and the operator (L,B) is reversible, then the operators $(L(k),B(k))$ are also reversible for large k.
2. If the boundary-value problem (1), (2) has a solution for any values of the right-hand sides, then the boundary-value problems (5), (6) also have a solution for large k.
3. If the boundary-value problem (1), (2) has a unique solution, then the problems (5), (6) also have a unique solution for each sufficiently large k.

Let us consider parameterized by number $\varepsilon\in[0,\varepsilon_0)$, $\varepsilon_0>0$, linear boundary-value problem

$$L(\varepsilon)y(t,\varepsilon):=y^{(r)}(t,\varepsilon)+\sum_{j=1}^{r}A_{r-j}(t,\varepsilon)y^{(r-j)}(t,\varepsilon)=f(t,\varepsilon),\ \ t\in(a,b), \tag{7}$$

$$B(\varepsilon)y(\,\cdot\,;\varepsilon)=c(\varepsilon), \tag{8}$$

where for every fixed ε the matrix-valued functions $A_{r-j}(\,\cdot\,;\varepsilon)\in(W_p^n)^{m\times m}$, the vector-valued function $f(\,\cdot\,;\varepsilon)\in(W_p^n)^m$, the vector $c(\varepsilon)\in\mathbb{C}^{rm}$, $B(\varepsilon)$ is the linear continuous operator $B(\varepsilon):(W_p^{n+r})^m\to\mathbb{C}^{rm}$, and the solution (the unknown vector-valued function) $y(\,\cdot\,;\varepsilon)\in(W_p^{n+r})^m$.

It follows from Theorem 1 that the boundary-value problem (7), (8) is a Fredholm one with index zero.

Definition 2. A solution to the boundary-value problem (7), (8) depends continuously on the parameter ε at $\varepsilon=0$ if the following two conditions are satisfied:

$(*)$ there exists a positive number $\varepsilon_1<\varepsilon_0$ such that for any $\varepsilon\in[0,\varepsilon_1)$ and arbitrary chosen right-hand sides $f(\,\cdot\,;\varepsilon)\in(W_p^n)^m$ and $c(\varepsilon)\in\mathbb{C}^{rm}$ this problem has a unique solution $y(\,\cdot\,;\varepsilon)$ that belongs to the space $(W_p^{n+r})^m$;

$(**)$ the convergence of the right-hand sides $f(\,\cdot\,;\varepsilon) \to f(\,\cdot\,;0)$ in $(W_p^n)^m$ and $c(\varepsilon) \to c(0)$ in $\mathbb{C}^{rm}$ as $\varepsilon \to 0+$ implies the convergence of the solutions $y(\,\cdot\,;\varepsilon) \to y(\,\cdot\,;0)$ in $(W_p^{n+r})^m$.

Consider the following conditions as $\varepsilon \to 0+$:

(0) the limiting homogeneous boundary-value problem

$$L(0)y(t,0) = 0, \;\; t \in (a,b), \;\; B(0)y(\,\cdot\,,0) = 0$$

has only the trivial solution;

(I) $A_{r-j}(\,\cdot\,;\varepsilon) \to A_{r-j}(\,\cdot\,;0)$ in the space $(W_p^n)^{m\times m}$ for each number $j \in \{1,\dots,r\}$;

(II) $B(\varepsilon)y \to B(0)y$ in the space $\mathbb{C}^{rm}$ for every $y \in (W_p^{n+r})^m$.

Theorem 4. *A solution to the boundary-value problem* (7), (8) *depends continuously on the parameter ε at $\varepsilon = 0$ if and only if this problem satisfies condition* (0) *and the conditions* (I) *and* (II).

We supplement our result with a two-sided estimate of the error $\|y(\,\cdot\,;0) - y(\,\cdot\,;\varepsilon)\|_{n+r,p}$ of the solution $y(\,\cdot\,;\varepsilon)$ via its discrepancy

$$\widetilde{d}_{n,p}(\varepsilon) := \big\|L(\varepsilon)y(\,\cdot\,;0) - f(\,\cdot\,;\varepsilon)\big\|_{n,p} + \big\|B(\varepsilon)y(\,\cdot\,;0) - c(\varepsilon)\big\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\,\cdot\,;0)$ as an approximate solution to problem (7), (8).

Theorem 5. *Suppose that the boundary-value problem* (7), (8) *satisfies conditions* (0), (I) *and* (II). *Then there exist positive numbers $\varepsilon_2 < \varepsilon_1$ and γ_1, γ_2 such that, for any $\varepsilon \in (0,\varepsilon_2)$, the following two-sided estimate is true:*

$$\gamma_1\,\widetilde{d}_{n,p}(\varepsilon) \le \big\|y(\,\cdot\,;0) - y(\,\cdot\,;\varepsilon)\big\|_{n+r,p} \le \gamma_2\,\widetilde{d}_{n,p}(\varepsilon),$$

where the quantities ε_2, γ_1, and γ_2 do not depend of $y(\,\cdot\,;\varepsilon)$ and $y(\,\cdot\,;0)$.

Thus, the error and discrepancy of the solution $y(\,\cdot\,;\varepsilon)$ to the boundary-value problem (7), (8) are of the same degree of smallness.

The results are published in [1, 3–7]. The most general class of multi-point boundary-value problems for systems of linear ordinary differential equations of an arbitrary order is considered in [2].

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Description of the Linear Perron Effect Under Parametric Perturbations Exponentially Decaying at Infinity

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1 Introduction

For a given integer $n \geq 2$ let $\mathcal{M}_n$ denote the class of linear differential systems

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1.1}$$

with continuous bounded coefficients defined on $\mathbb{R}_+$. Let us denote by $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ the Lyapunov exponents [7, p. 561], [1, p. 38] of system (1.1), by $\Lambda(A) = (\lambda_1(A), \dots, \lambda_n(A))$ their spectrum, and by $\mathrm{es}(A)$ its exponential stability index (i.e. the dimension of the linear subspace of solutions to this system that have negative characteristic exponents). In what follows, we identify system (1.1) with its defining function $A(\,\cdot\,)$ and therefore write $A \in \mathcal{M}_n$.

In his seminal paper [10] O. Perron constructed an example of a system $A \in \mathcal{M}_2$ for which there exists an exponentially decaying perturbation $Q : \mathbb{R}_+ \to \mathbb{R}^{2\times 2}$ such that the perturbed system

$$\dot{x} = (A(t) + Q(t))x, \;\; x \in \mathbb{R}^2, \;\; t \in \mathbb{R}_+,$$

has the Lyapunov exponents

$$\lambda_2(A + Q) > \lambda_2(A) \;\text{ and }\; \lambda_1(A + Q) = \lambda_1(A). \tag{1.2}$$

The largest Lyapunov exponent $\lambda_2(A)$ of the unperturbed system A in Perron's example is positive, and hence this system is unstable and so is the perturbed one. In fact, the same example can be slightly modified to demonstrate the phenomenon of loss of stability in a linear system under exponentially decaying perturbation of its coefficients. Let $\sigma \in (\lambda_2(A), \lambda_2(A + Q))$. Then for the modified system $\widetilde{A} \equiv A - \sigma I_2$, where I_2 is the (2×2) identity matrix, we have

$$\mathrm{es}(\widetilde{A}) = 2 \;\text{ and }\; \mathrm{es}(\widetilde{A} + Q) = 1.$$

Thus, the system $\widetilde{A}$ is exponentially stable, whereas the perturbed system $\widetilde{A}+Q$ is only conditionally exponentially stable.

O. Perron also constructed [9] an example of a system $A \in \mathcal{M}_2$ with negative Laypunov exponents and its quadratic perturbation $f(x)$ such that the perturbed system $\dot{x} = A(t)x + f(x)$ possesses the following property: the characteristic exponent of any nontrivial solution starting at the line $x_1 = 0$ is the same as for the unperturbed system, while the characteristic exponent of any solution starting outside the line $x_1 = 0$ is greater than a certain positive number.

These examples by Perron served as a starting point for numerous studies of the effect of various classes of linear and nonliner perturbations on the Lyapunov exponents of systems in $\mathcal{M}_n$. The results obtained in this direction constitute an essential part of the modern theory of Lyapunov exponents. The effect of change of Lyapunov exponents of a system in $\mathcal{M}_n$ under one or another "small" perturbation was called in the monograph [6, Ch. 4] the Perron effect. Starting with the paper [5], the term is being used only for situations when perturbations do not decrease the Lyapunov exponents of the original system (in what follows, we will adhere to this terminology). Unlike the papers [5,6], which study the Perron effect under higher-order perturbations, and along the lines of the paper [10] we investigate linear vanishing at infinity perturbations of the coefficient matrix of a system in $\mathcal{M}_n$ and call this effect "linear Perron".

It is worth noting that the perturbation matrix constructed in paper [10] is of the form $Q(t) = \mu Q_0(t)$, where μ is a real parameter; it is established there that for each $\mu \neq 0$ relations(1.2) hold. With this in mind, given a metric space M we consider families of linear systems of the form

$$\dot{x} = (A(t) + Q(t,\mu))x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+, \tag{1.3}$$

where $A \in \mathcal{M}_n$ and $Q(\,\cdot\,,\,\cdot\,) : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ is jointly continuous matrix-valued function. For each fixed value of the parameter $\mu \in M$ we get a linear differential system with continuous bounded coefficients whose Lyapunov exponents will be denoted by $\lambda_1(\mu; A+Q) \leqslant \cdots \leqslant \lambda_n(\mu; A+Q)$. Therefore, the Lyapunov exponents of family (1.3) are functions of the parameter $\mu \in M$. In particular, the spectrum of family(1.3) is defined to be the vector function $\Lambda(\,\cdot\,; A+Q) \equiv (\lambda_1(\,\cdot\,; A+Q), \dots, \lambda_n(\,\cdot\,; A+Q)) : M \to \mathbb{R}^n$.

2 Statement of the problem. Main result

We will denote by $\mathcal{E}_n(M)$ the class of jointly continuous matrix-valued functions $Q : \mathbb{R}_+ \times M \to \mathbb{R}^{n\times n}$ satisfying the estimate

$$\|Q(t,\mu)\| \leqslant C_Q \exp(-\sigma_Q t), \quad (t,\mu) \in \mathbb{R}_+ \times M,$$

with C_Q and σ_Q being positive constants (different for each function Q).

For a system $A \in \mathcal{M}_n$ we will denote by $\mathcal{E}_n[A](M)$ the class of those $Q \in \mathcal{E}_n(M)$ that do not descrease its Lyapunov exponents, i.e. for any $A \in \mathcal{M}_n$ and its perturbation $Q \in \mathcal{E}_n[A](M)$ the inequalities

$$\inf_{\mu\in M} \lambda_i(\mu; A+Q) \geqslant \lambda_i(A), \quad i = 1, \dots, n,$$

hold. Clearly, for any $A \in \mathcal{M}_n$ the class $\mathcal{E}_n[A](M)$ is nonempty since identically zero matrix belongs to it.

The problem to be solved is to obtain for each $n \geq 2$ and each metric space M a complete description of the class of pairs $(\Lambda(A), \Lambda(\,\cdot\,; A+Q))$ composed of the spectrum $\Lambda(A) \in \mathbb{R}^n$ of a system $A \in \mathcal{M}_n$ and the spectrum $\Lambda(\,\cdot\,; A+Q) : M \to \mathbb{R}^n$ of a family $A+Q$, where A ranges over $\mathcal{M}_n$ and matrix-valued function Q ranges over the class $\mathcal{E}_n[A](M)$ for each A, i.e. of the class

$$\Pi\mathcal{E}_n(M) = \Big\{(\Lambda(A), \Lambda(\,\cdot\,; A+Q)) \mid \ A \in \mathcal{M}_n, \ \ Q \in \mathcal{E}_n[A](M)\Big\}.$$

Note that a complete description of the class

$$\Lambda\mathcal{E}_n(M) = \Big\{\Lambda(\,\cdot\,; A+Q) \mid \ A \in \mathcal{M}_n, \ \ Q \in \mathcal{E}_n[A](M)\Big\},$$

composed of the second elements of the pairs in the class $\Pi\mathcal{E}_n(M)$ immediately follows from the result of [3].

Obviously, the solution of the stated problem would contain as a special case Perron's example and describe from a descriptive set theoretic standpoint all possible situations in which an exponentially stable linear system gets unstable under parametric exponentially vanishing perturbations. For instance, it follows from the theorem presented below that there exists a system $A \in \mathcal{M}_2$ with the largest Laypunov exponent $\lambda_2(A) = -1$ and its perturbation $Q \in \mathcal{E}_2[A](\mathbb{R})$ such that the largest Laypunov exponent $\lambda_2(A+Q)$ of the perturbed system equals -1 for a rational μ and 1 for an irrational μ.

The direction in the theory of Lyapunov exponents dealing with the dependence of asymptotic properties and characteristics of parametric differential systems on the parameter is due to V. M. Millionshchikov, who initiated systematic research in this direction with a series of papers, of which we only mention the paper [8]. We are also indebted to him for understanding that the language of the Baire theory of discontinuous functions is adequate for describing such a dependence. We emphasize that here one speaks of a complete description of all possible types of behavior of some properties or characteristics of a system under changes in the system parameters as opposed to establishing sufficient conditions for one or another type of their behavior. Since then, quite a few results have been obtained in this vein.

Let us recall that a function $f : M \to \mathbb{R}$ is said [4, pp. 266–267] to be of the class $({}^*, G_\delta)$ if for each $r \in \mathbb{R}$ the preimage $f^{-1}([r, +\infty))$ of the half-interval $[r, +\infty)$ is a G_δ-set of the metric space M. In particular, the class $({}^*, G_\delta)$ is a subclass of the second Baire class [4, p. 294].

A complete description of the class $\Pi\mathcal{E}_n(M)$ for any $n \geq 2$ and metric space M is given by the following statement [2].

Theorem. *Let $n \geqslant 2$ be an integer and M a metric space. A pair $(l, F(\,\cdot\,))$, with $l = (l_1, \dots, l_n) \in \mathbb{R}^n$ and $F(\,\cdot\,) = (f_1(\,\cdot\,), \dots, f_n(\,\cdot\,)) : M \to \mathbb{R}^n$, belongs to the class $\Pi\mathcal{E}_n(M)$ if and only if the following conditions are met:*

(1) $l_1 \leqslant \cdots \leqslant l_n$;

(2) $f_1(\mu) \leqslant \cdots \leqslant f_n(\mu)$ *for all* $\mu \in M$;

(3) $f_i(\mu) \geqslant l_i$ *for all* $\mu \in M$ *and* $i = 1, \dots, n$;

(4) *for each* $i = 1, \dots, n$ *the function* $f_i(\,\cdot\,) : M \to \mathbb{R}$ *is bounded and is of the class* $({}^*, G_\delta)$.

Corollary 2.1. *Let $n \geqslant 2$ be an integer and M an interval in the real line. Then for each pair $(l, F(\,\cdot\,)) \in \Pi\mathcal{E}_n(M)$ there exists a system $A \in \mathcal{M}_n$ and its perturbation $Q \in \mathcal{E}_n[A](\mathbb{R})$ analytical in parameter such that $\Lambda(A) = l$ and $\Lambda(\,\cdot\,; A+Q) = F$.*

Let $\mathcal{Z}_n \equiv \{0, \dots, n\}$. We define the function $\operatorname{es}(\,\cdot\,; A) : M \to \mathcal{Z}_n$ assigning to each $\mu \in M$ the exponential stability index of system (1.3). There naturally arises the problem of describing the class of pairs composed of the exponential stability index $\operatorname{es}(A) \in \mathcal{Z}_n$ of a system A and the exponential stability index $\operatorname{es}(\,\cdot\,; A+Q) : M \to \mathcal{Z}_n$ of a family $A+Q$, i.e. of the class

$$\mathcal{I}\mathcal{E}_n(M) = \Big\{ (\operatorname{es}(A), \operatorname{es}(\,\cdot\,; A+Q)) \mid \ A \in \mathcal{M}_n, \ \ Q \in \mathcal{E}_n[A](M) \Big\}.$$

The solution is provided by the following statement.

Corollary 2.2. *Let $n \geqslant 2$ be an integer and M a metric space. A pair $(d, f(\,\cdot\,))$, where $d \in \mathcal{Z}_n$ and $f : M \to \mathcal{Z}_n$, belongs to the class $\mathcal{I}\mathcal{E}_n(M)$ if and only if $f(\mu) \leqslant d$ for all $\mu \in M$ and the function $(-f)$ is of the class $({}^*, G_\delta)$.*

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On Solvability of Poisson's Equation with Mixed Dirichlet and Nonlocal Integral Type Conditions

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Abstract. In this paper we study a problem for Poisson's equation when on opposite sides of rectangular domain are given the Dirichlet conditions, while on the rest two sides are given integral type nonlocal constraints. We prove the existence and uniqueness of a solution in the weighted Sobolev space.

Let $\Omega=\{x=(x_1,x_2):\ 0<x_k<l,\ k=1,2\}$ be a square with boundary Γ.
We seek in Ω a solution to Poisson's equation

$$\Delta u=-f(x),\ \ x\in\Omega, \tag{1}$$

which satisfies the following Dirichlet homogeneous conditions

$$u(x_1,0)=u(x_1,l)=0,\ \ 0\le x_1\le l, \tag{2}$$

and the integral type nonlocal conditions

$$\int\limits_0^\xi u(x)\,dx_1=0,\quad \int\limits_{l-\xi}^l u(x)\,dx_1=0,\ \ 0\le x_2\le l,\ \ 0<\xi\le\frac{l}{2}\,. \tag{3}$$

By $L_2(\Omega,\rho)$ we denote a weighted Lebesgue space of all real-valued functions $u(x)$ on Ω with the inner product and the norm

$$(u,v)_\rho=\int\limits_\Omega \rho uv\,dx,\ \ \|u\|_\rho=(u,u)_\rho^{1/2}.$$

Denote by $\overset{*}{W}{}_2^1(\Omega,\rho)$ a linear set of all functions $L_2(\Omega,\rho)$ whose first order derivatives (in general sense) belong to $L_2(\Omega,\rho)$. It is a normalized space with the norm

$$\|u\|_{1,\rho}=\big(\|u\|_\rho^2+|u|_{1,\rho}^2\big)^{1/2},\ \ |u|_{1,\rho}^2=\Big\|\frac{\partial u}{\partial x_1}\Big\|_\rho^2+\Big\|\frac{\partial u}{\partial x_2}\Big\|_\rho^2.$$

Let us choose a weight function $\rho(x)$ in the following form

$$\rho(x):=\begin{cases}\dfrac{x_1}{\xi}, & 0\le x_1\le\xi,\\ 1, & \xi<x_1<l-\xi,\\ \dfrac{l-x_1}{\xi}, & l-\xi\le x_1\le l,\end{cases}$$

and define an operator in the form

$$Gv(x):=\begin{cases}\dfrac{x_1}{\xi}\,v(x)-\dfrac{1}{\xi}\displaystyle\int_0^{x_1}v(t,x_2)\,dt, & 0\le x_1\le\xi,\\ v(x), & \xi<x_1<l-\xi,\\ \dfrac{l-x_1}{\xi}\,v(x)-\dfrac{1}{\xi}\displaystyle\int_{x_1}^{l}v(t,x_2)\,dt, & l-\xi\le x_1\le l.\end{cases}$$

We say that function $u\in\overset{*}{W}{}_2^1(\Omega,\rho)$ is a weak solution of problem (1)–(3) if the relation

$$a(u,v)=(f,Gv),\quad \forall\,v\in\overset{*}{W}{}_2^1(\Omega,\rho) \tag{4}$$

holds, where

$$a(u,v):=\Big(\frac{\partial u}{\partial x_1}\,,\frac{\partial v}{\partial x_1}\Big)_\rho+\Big(\frac{\partial u}{\partial x_2}\,,G\,\frac{\partial v}{\partial x_2}\Big). \tag{5}$$

Equation (4) can be formally obtained from (1) by taking into account conditions (2), (3).

In the case $u=v$ for estimate of the second addend of (5) we use the following proposition.

Lemma 1. *If the function v defined on the segment [0;1] satisfies the nonlocal conditions* (3)*, then the following identity*

$$\int_0^l v(x)Gv(x)\,dx_1=\int_0^l\rho(x)v^2(x)\,dx_1$$

holds.

Indeed, $v=\frac{\partial u}{\partial x_2}$ satisfies the nonlocal conditions. Besides, we take into account the equalities implied from the definition of the operator G

$$\int_0^\xi v(x_1,x_2)\int_0^{x_1}v(t,x_2)\;dt\,dx_1=\frac12\bigg(\int_0^{x_1}v(t,x_2)\,dt\bigg)^2\bigg|_{x_1=0}^{\xi}=0,$$

$$\int_{l-\xi}^{l} v(x_1,x_2)\int_{x_1}^{l}v(t,x_2)\;dt\,dx_1=-\frac12\bigg(\int_{x_1}^{l}v(t,x_2)\,dt\bigg)^2\bigg|_{x_1=l-\xi}^{l}=0.$$

In view of the following equalities

$$\frac{\partial}{\partial x_1}\,(Gv)=\rho\,\frac{\partial v}{\partial x_1}\,, \quad \int\limits_{\Omega}\frac{\partial^2 u}{\partial x_1^2}\,Gv\,dx_1\,dx_2=-\int\limits_{\Omega}\rho\,\frac{\partial u}{\partial x_1}\,\frac{\partial v}{\partial x_1}\,dx_1\,dx_2,$$

one can prove that the bilinear form $a(u,v)$ is both continuous and coercive on $u\in \overset{*}{W}{}_2^1(\Omega,\rho)$, while the linear form (f,Gv) is continuous on the same space.

Lemma 2. *For any function $u\in \overset{*}{W}{}_2^1(\Omega,\rho)$, the estimate*

$$\int\limits_0^l u^2\,dx_1\le\frac{5l^2}{4}\int\limits_0^l \rho\Big(\frac{\partial u}{\partial x_1}\Big)^2\,dx_1$$

is valid.

Proof. For simplicity let us write u' instead of $\partial u/\partial x_1$. Let

$$J:=-\int\limits_0^{\xi} x_1\,du^2+\int\limits_{\xi}^{l-\xi}\Big(\frac{l}{2}-x_1\Big)\,du^2+\int\limits_{l-\xi}^{l}(l-x_1)\,du^2. \tag{6}$$

It is easy to verify that

$$\int\limits_0^l u^2\,dx_1=J+\frac{l}{2}\,\big[u^2(\xi)+u^2(l-\xi)\big]. \tag{7}$$

Rewrite (6) as follows

$$J=-2\int\limits_0^{\xi} x_1u'u\,dx_1+2\int\limits_{\xi}^{l-\xi}\Big(\frac{l}{2}-x_1\Big)\,u'u\,dx_1+2\int\limits_{l-\xi}^{l}(l-x_1)u'u\,dx_1.$$

Whence, by use of ε-inequality, we obtain

$$\begin{aligned}|J|\le\bigg[\frac12\int\limits_0^{\xi}u^2\,dx_1+2\int\limits_0^{\xi}x_1^2(u')^2\,dx_1\bigg]+\bigg[\frac12\int\limits_{\xi}^{l-\xi}u^2\,dx_1+2\int\limits_{\xi}^{l-\xi}\Big(\frac{l}{2}-x_1\Big)^2(u')^2\,dx_1\bigg]\\ +\bigg[\frac12\int\limits_{l-\xi}^{l}u^2\,dx_1+2\int\limits_{l-\xi}^{l}(l-x_1)^2(u')^2\,dx_1\bigg].\end{aligned}$$

Now let us estimate the values $u(\xi)$, $u(l-\xi)$,

$$|\xi u(l-\xi)|^2=\bigg(\int\limits_{l-\xi}^{l}(l-x_1)u'\,dx_1\bigg)^2\le\frac{\xi^2}{2}\int\limits_{l-\xi}^{l}(l-x_1)(u')^2\,dx_1,$$

$$u^2(l-\xi)\le\frac12\int\limits_{l-\xi}^{l}(l-x_1)(u')^2\,dx_1;\quad u^2(\xi)\le\frac12\int\limits_0^{\xi}x_1(u')^2\,dx_1.$$

Finally, from (7) it follows

$$
\begin{aligned}
\frac{1}{2}\int\limits_0^l u^2\,dx_1 \le 2\int\limits_0^{\xi} x_1^2(u')^2\,dx_1 + 2\int\limits_{\xi}^{l-\xi}\Big(\frac{l}{2}-x_1\Big)^2(u')^2\,dx_1 + 2\int\limits_{l-\xi}^{l}(l-x_1)^2(u')^2\,dx_1 \\
+\frac{l}{4}\int\limits_0^{\xi} x_1(u')^2\,dx_1 + \frac{l}{4}\int\limits_{l-\xi}^{l}(l-x_1)(u')^2\,dx_1,
\end{aligned}
$$

which confirms Lemma 2.

Thus, all the conditions of the Lax–Milgram lemma are fulfilled. Therefore, problem (1)–(3) has a unique weak solution from $\overset{*}{W}{}_2^1(\Omega,\rho)$.

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Asymptotic Properties of Special Classes of Solutions of Second-Order Differential Equations with Nonlinearities Near to Regularly Varying

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The differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y')\exp\big(R(|\ln|yy'||)\big), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ are continuous functions, $Y_i \in \{0, \pm\infty\}$ $(i = 0, 1)$, Δ_{Y_i} is a one-sided neighborhood of Y_i, every function $\varphi_i(z)$ $(i = 0, 1)$ is a regularly varying function as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of order σ_i, $\sigma_0 + \sigma_1 \ne 1$, $\sigma_1 \ne 0$, the function $R :]0, +\infty[\to]0, +\infty[$ is continuously differentiable and regularly varying on infinity of the order μ, $0 < \mu < 1$, the derivative function of the function R is monotone, is considered in the work.

Definition. A solution y of equation (1) is called $P_\omega(Y_0, Y_1, \lambda_0)$ if it is defined on $[t_0, \omega[\subset [a, \omega[$ and

$$\lim_{t\uparrow\omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t\uparrow\omega} \frac{(y'(t))^2}{y(t)y''(t)} = \lambda_0.$$

A lot of works (see, for example, [3, 4]) have been devoted to the establishing asymptotic representations of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equations of the form (1), in which $R \equiv 0$. The $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) are regularly varying functions as $t \uparrow \omega$ of index $\frac{\lambda_0}{\lambda_0 - 1}$ if $\lambda_0 \in R \setminus \{0, 1\}$. The asymptotic properties and necessary and sufficient conditions of existence of such solutions of equation (1) have been obtained in [1].

The case $\lambda_0 = \infty$ is one of the most difficult cases because in this case such solutions or their derivatives are slowly varying functions as $t \uparrow \omega$. Some results about asymptotic properties and existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) in the special case are presented in the work.

We say that a slowly varying as $z \to Y$ $(z \in \Delta_Y)$ function $\theta : \Delta_Y \to]0; +\infty[$ satisfies the condition S if for any continuous differentiable function $L : \Delta_{Y_i} \to]0; +\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}} \frac{zL'(z)}{L(z)} = 0,$$

the next equality

$$\Theta(zL(z)) = \Theta(z)(1 + o(1)) \ \text{ is true as } \ z \to Y \ \ (z \in \Delta_Y)$$

holds.

Let us introduce the following notations

$$\pi_\omega(t) = \begin{cases} t & \text{as } \ \omega = +\infty, \\ t - \omega & \text{as } \ \omega < +\infty, \end{cases} \quad \Theta_i(z) = \varphi_i(z)|z|^{-\sigma_i} \ \ (i = 0, 1),$$

$$I(t)=\alpha_0\int\limits_{A_\omega}^{t}p(\tau)\,d\tau,\quad A_\omega=\begin{cases}a & \text{if } \int\limits_a^\omega p(\tau)\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_a^\omega p(\tau)\,d\tau<+\infty.\end{cases}$$

In case $\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|\operatorname{sign}y_0^0=Y_0$ we put

$$I_0(t)=\alpha_0\int\limits_{A_\omega^0}^{t}p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\Theta_0\big(|\pi_\omega(\tau)|\operatorname{sign}y_0^0\big)\,d\tau,$$

$$A_\omega^0=\begin{cases}b_2 & \text{if } \int\limits_{b_2}^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|\operatorname{sign}y_0^0\big)\,dt=+\infty,\\ \omega & \text{if } \int\limits_{b_2}^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\Theta_0\big(|\pi_\omega(t)|y_0^0\big)\,dt<+\infty,\end{cases}$$

$$N(t)=\alpha_0p(t)|\pi_\omega(t)|^{\sigma_0+1}\Theta_0\big(|\pi_\omega(t)|\operatorname{sign}y_0^0\big).$$

Here $b_1,b_2\in[a;\omega[$ are chosen in such a way that $\frac{\operatorname{sign}y_0^1}{\|\pi_\omega(t)|}\in\Delta_{Y_1}$ as $t\in[b_1;\omega]$ and $|\pi_\omega(\tau)|\operatorname{sign}y_0^0\in\Delta_{Y_0}$ as $t\in[b_2;\omega]$.

The next three theorems are devoted to establishing $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of equation (1). First two cases are obtained in [2]. The first derivatives of such solutions are slowly varying functions as $t\uparrow\omega$, the fact creates difficulties in investigation of such solutions.

Theorem 1. *For the existence of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of equation* (1) *the following conditions are necessary*

$$Y_0=\begin{cases}\pm\infty & \text{if } \omega=+\infty,\\ 0 & \text{if } \omega<+\infty,\end{cases}\qquad \pi_\omega(t)y_0^0y_1^0>0\ \text{ as }\ t\in[a,\omega[\,. \tag{2}$$

If the function φ_0 satisfies the condition S and

$$\lim_{t\uparrow\omega}\frac{R'(|\ln|\pi_\omega(t)||)I_0(t)}{\pi_\omega(t)I_0'(t)}=0, \tag{3}$$

then (2) *together with the next conditions are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of equation* (1)*:*

$$\lim_{t\uparrow\omega}y_1^0|I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}=Y_1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)}{I_0(t)}=0,\quad y_1^0(1-\sigma_0-\sigma_1)I_0(t)>0\ \text{ as }\ t\in[b_2,\omega[\,.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))}=(1-\sigma_0-\sigma_1)I_0(t)[1+o(1)],\quad \frac{y'(t)}{y(t)}=\frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

Theorem 2. *If in* (1) *the function p is a continuously differentiable function, the function φ_0 satisfies the condition S and*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)}=0, \tag{4}$$

then together with (2) *the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions of equation* (1)*:*

$$\lim_{t\uparrow\omega} y_1^0 \exp\Big(\frac{1}{1-\sigma_0-\sigma_1}\, R\big(|\ln|\pi_\omega(t)||\big)\Big) = Y_1, \quad \alpha_0 y_1^0(1-\sigma_0-\sigma_1)\ln|\pi_\omega(t)| > 0 \ \ as \ \ t\in[a,\omega[\,.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$

$$\frac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)y'(t)||))} = \frac{|1-\sigma_0-\sigma_1|N(t)}{R'(|\ln|\pi_\omega(t)||)}\,[1+o(1)], \quad \frac{y'(t)}{y(t)} = \frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

Theorem 3. *If in* (1) *the function* p *is a continuously differentiable function, the function* φ_0 *satisfies the condition* S *and*

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)N'(t)}{R'(|\ln|\pi_\omega(t)||)N(t)} = M \neq 0, \tag{5}$$

then together with (2) *the following conditions are necessary and sufficient for the existence of* $P_\omega(Y_0, Y_1, \pm\infty)$*-solutions of equation* (1)*:*

$$\lim_{t\uparrow\omega} y_1^0 \exp\Big(\frac{1}{1-\sigma_0-\sigma_1} R(|\ln|\pi_\omega(t)||)\Big) = Y_1, \ \ \alpha_0 y_1^0(M+1)(1-\sigma_0-\sigma_1)\ln|\pi_\omega(t)| > 0 \ \ as \ \ t\in[a,\omega[\,.$$

For such solutions the next asymptotic representations take place as $t\uparrow\omega$

$$\begin{aligned}\frac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)y'(t)||))} &= \frac{|1-\sigma_0-\sigma_1|N(t)(M+1)}{R'(|\ln|\pi_\omega(t)||)}\,[1+o(1)],\\ \frac{y'(t)}{y(t)} &= \frac{1}{\pi_\omega(t)}\,[1+o(1)].\end{aligned}$$

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Boundary-Value Problems for Weakly Singular Integral Equations

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In the space $L_2[a,b]$, we consider a perturbed linear boundary-value problem for a weakly singular integral equation

$$x(t)-\int_a^b K(t,s)x(s)\,ds=f(t)+\varepsilon\int_a^b \overline{K}(t,s)x(s)\,ds, \tag{1}$$

$$lx(\,\cdot\,)=\alpha+\varepsilon Jx(\,\cdot\,). \tag{2}$$

We establish conditions for the bifurcation of solutions of the boundary-value problem (1), (2) and determine the structure of these solutions under the condition that the generating boundary-value problem

$$x(t)-\int_a^b K(t,s)x(s)\,ds=f(t),\ \ lx(\,\cdot\,)=\alpha \tag{3}$$

is unsolvable.

Here, $K(t,s)=\frac{H(t,s)}{|t-s|^\gamma}$ and $\overline{K}(t,s)=\frac{\overline{H}(t,s)}{|t-s|^\beta}$, where $H(t,s)$, $\overline{H}(t,s)$ are functions bounded in the domain $[a,b]\times[a,b]$, $0<\gamma<1$, $0<\beta<1$, $f\in L_2[a,b]$, $l=col(l_1,l_2,\dots,l_p):L_2[a,b]\to\mathbb{R}^p$, $J=col(J_1,J_2,\dots,J_p):L_2[a,b]\to\mathbb{R}^p$ are bounded linear functionals, $l_\nu,J_\nu:L_2[a,b]\to\mathbb{R}$, $\nu=\overline{1,p}$, $\alpha=col(\alpha_1,\alpha_2,\dots,\alpha_p)\in\mathbb{R}^p$ and $\varepsilon\ll 1$ is a small parameter.

By using the results obtained in [2], we show that the study of the problem of appearance of solutions of the boundary-value problem (1), (2) reduces to the corresponding task for the perturbed boundary-value problem for the Fredholm integral equation

$$x(t)=f_n(t)+\sum_{k=0}^{n}\varepsilon^k\int_a^b R_n^k(t,s)x(s)\,ds, \tag{4}$$

$$f_n(t)=f(t)+\sum_{k=1}^{n-1}\int_a^b R_k^0(t,s)f(s)\,ds+\sum_{k=1}^{n-1}\varepsilon^k\sum_{m=k}^{n-1}\int_a^b R_m^k(t,s)f(s)\,ds,$$

where $R_n^k(t,s)$, $k=\overline{0,n}$, are the sums of C_n^k kernels of all possible products of $n-k$ integral operators K and k integral operators $\overline{K}$

$$(Kw)(t)=\int_a^b\frac{H(t,s)}{|t-s|^\gamma}\,w(s)\,ds\ \text{ and }\ (\overline{K}w)(t)=\int_a^b\frac{\overline{H}(t,s)}{|t-s|^\beta}\,w(s)\,ds.$$

We apply the approach described in [3] to the study of the boundary-value problem (4), (2) and show that it can be reduced to the operator equation. Let $\{\varphi_i(t)\}_{i=1}^\infty$ be a complete orthonormal system of functions in $L_2[a,b]$. We introduce the notation

$$x_i=\int\limits_a^b x(t)\varphi_i(t)\,dt, \quad a_{ij}=\int\limits_a^b\int\limits_a^b K_n(t,s)\varphi_i(t)\varphi_j(s)\,dt\,ds,$$

$$a_{ij}^k=\int\limits_a^b\int\limits_a^b R_n^k(t,s)\varphi_i(t)\varphi_j(s)\,dt\,ds, \quad k=\overline{1,n},$$

$$f_i=\int\limits_a^b f(t)\varphi_i(t)\,dt+\sum_{k=1}^{n-1}\int\limits_a^b\int\limits_a^b R_k^0(t,s)f(s)\varphi_i(t)\,dt\,ds,$$

$$f_i^k=\sum_{m=k}^{n-1}\int\limits_a^b\int\limits_a^b R_m^k(t,s)f(s)\varphi_i(t)\,ds\,dt, \quad k=\overline{1,n-1}.$$

By using this notation in the boundary-value problem (4), (2), we obtain the operator equation:

$$Uz=q+\sum_{k=1}^{n-1}\varepsilon^k q_k+\sum_{k=1}^{n}\varepsilon^k U_k z, \tag{5}$$

where

$$U=\begin{bmatrix}\Lambda\\ W\end{bmatrix}, \quad U_1=\begin{bmatrix}\Lambda_1\\ W_1\end{bmatrix}, \quad U_k=\begin{bmatrix}\Lambda_k\\ 0\end{bmatrix}, \quad k=\overline{2,n},$$

$$q=\begin{bmatrix}g\\ \alpha\end{bmatrix}, \quad q_k=\begin{bmatrix}g_k\\ 0\end{bmatrix}, \quad k=\overline{1,n-1},$$

where the vectors z, g, g_k, $k=\overline{1,n-1}$ and the matrices W, W_1, Λ, Λ_k, $k=\overline{1,n}$ have the form

$$z=col\big(x_1, \ \ x_2, \ \ \dots, \ \ x_i, \ \ \dots\big), \quad g=col\big(f_1, \ \ f_2, \ \ \dots, \ \ f_i, \ \ \dots\big),$$

$$g_k=col\big(f_1^k, \ \ f_2^k, \ \ \dots, \ \ f_i^k, \ \ \dots\big), \quad W=l\Phi(\,\cdot\,), \quad W_1=J\Phi(\,\cdot\,),$$

$$\Lambda=\begin{pmatrix}1-a_{11} & -a_{12} & \dots & -a_{1i} & \dots\\ -a_{21} & 1-a_{22} & \dots & -a_{2i} & \dots\\ \dots & \dots & \dots & \dots & \dots\\ -a_{i1} & -a_{i2} & \dots & 1-a_{ii} & \dots\\ \dots & \dots & \dots & \dots & \dots\end{pmatrix}, \quad \Lambda_k=\begin{pmatrix}a_{11}^k & a_{12}^k & \dots & a_{1i}^k & \dots\\ a_{21}^k & a_{22}^k & \dots & a_{2i}^k & \dots\\ \dots & \dots & \dots & \dots & \dots\\ a_{i1}^k & a_{i2}^k & \dots & a_{ii}^k & \dots\\ \dots & \dots & \dots & \dots & \dots\end{pmatrix},$$

$$\Phi(t)=\big(\varphi_1(t),\varphi_2(t),\dots,\varphi_i(t),\dots\big).$$

The generating equation for the operator equation (5) has the form

$$Uz=q. \tag{6}$$

The operator $\Lambda:\ell_2\to\ell_2$ appearing on the left-hand side of the operator equation (6) has the form $\Lambda=I-A$, where $I:\ell_2\to\ell_2$ is the identity operator and $A:\ell_2\to\ell_2$ is a compact operator. Hence, according to S. Krein's classification, the operator $\Lambda:\ell_2\to\ell_2$ is a Fredholm operator of index zero ($\dim\ker\Lambda=\dim\ker\Lambda^*<\infty$) and the operator $U:\ell_2\to\ell_2\times\mathbb{R}^p$ is a Fredholm operator of nonzero index ($\dim\ker U<\infty$, $\dim\ker U^*<\infty$).

The following statement is true for equation (6) (see [4]).

Theorem 1. *The homogeneous equation* (6) ($q=0$) *possesses a* d_2*-parameter family of solutions* $z\in\ell_2$,

$$z=P_{\Lambda_r}P_{Q_{d_2}}c_{d_2}\ \ \forall\,c_{d_2}\in\mathbb{R}^{d_2}.$$

The inhomogeneous equation (6) *is solvable if and only if the following* $r+d_1$ *linearly independent conditions are satisfied:*

$$P_{\Lambda_r^*}g=0,\quad P_{Q_{d_1}^*}(\alpha-W\Lambda^+g)=0,$$

and the equation possesses a d_2*-parameter family of solutions* $z\in\ell_2$ *of the form*

$$z=P_{\Lambda_r}P_{Q_{d_2}}c_{d_2}+P_{\Lambda_r}Q^+(\alpha-W\Lambda^+g)+\Lambda^+g\quad\forall c_{d_2}\in\mathbb{R}^{d_2}.$$

Here, $Q=WP_{\Lambda_r}$ is a $(p\times r)$-matrix, $P_{\Lambda_r}(P_{\Lambda_r^*})$ is a matrix formed by a complete system of r linearly independent columns (rows) of the matrix projector P_Λ (P_{Λ^*}), where P_Λ (P_{Λ^*}) is the projector onto the kernel (cokernel) of the matrix Λ, and $P_{Q_{d_2}}$ ($P_{Q_{d_1}^*}$) is a matrix formed by the complete system of d_2 (d_1) linearly independent columns (rows) of the matrix projector P_Q (P_{Q^*}), where P_Q (P_{Q^*}) is the projector onto the kernel (cokernel) of the matrix Q and Λ^+ (Q^+) is the pseudoinverse Moore–Penrose matrix for the matrix Λ (Q).

We now determine the conditions required for the bifurcation of solutions of the perturbed inhomogeneous boundary-value problem (1), (2) and study the structure of these solutions under the conditions that the solution of the homogeneous generating the boundary-value problem (3) ($f(t)=0$, $\alpha=0$) is not unique, i.e. (see [2]), $P_{\Lambda_r}P_{Q_{d_2}}\neq0$, and that the inhomogeneous generating boundary-value problem (3) is unsolvable.

It is known (see [9]) that small perturbations preserve the Fredholm property of the operator, i.e., the operator $\big(U-\sum\limits_{k=1}^{n}\varepsilon^kU_k\big)$ is a Fredholm operator with nonzero index. This enables one to investigate equation (5) by the methods of the theory of perturbed operator boundary-value problems with Fredholm linear part (see, e.g., [1, 4, 11]) obtained as a generalization of the classical methods of the perturbation theory of periodic boundary-value problems in the theory of oscillations (see [5, 7, 8, 10]).

The analysis of the appearance of solutions of equation (5) is closely connected with the $((r+d_1)\times d_2)$-matrix

$$B_0=\begin{bmatrix}P_{\Lambda_r^*}\Lambda_1P_{\Lambda_r}P_{Q_{d_2}}\\ P_{Q_{d_1}^*}(W_1-W\Lambda^+\Lambda_1)P_{\Lambda_r}P_{Q_{d_2}}\end{bmatrix},$$

constructed by using the coefficients of equation (5).

We introduce an $((r+d_1)\times(r+d_1))$-matrix $P_{B_0^*}$, which is a projector onto the cokernel of the matrix B_0 and a matrix

$$G=\begin{bmatrix}-P_{\Lambda_r^*} & 0\\ P_{Q_{d_1}^*}W\Lambda^+ & -P_{Q_{d_1}^*}\end{bmatrix},$$

formed by $r+d_1$ rows and infinitely many columns. Moreover, as the matrix B_0, it is completely determined by the coefficients of equation (5).

By the Vishik–Lyusternik method (see [12]), we find efficient conditions for the coefficients guaranteeing the appearance of a family of solutions of the perturbed linear boundary-value problem (5) in the form of a Laurent series in powers of the small parameter ε with singularity at the point $\varepsilon=0$.

The results obtained for the perturbed equations (5) enable us to establish the conditions for the existence of a d_2-parameter family of solutions of the original perturbed boundary-value problem (1), (2). Indeed, if the boundary-value problem (1), (2) possesses at least one solution, then, according to the Riesz–Fischer theorem, one can find an element $x \in L_2[a,b]$ such that the quantities x_i, $i = \overline{1,\infty}$, determined from equation (5) are the Fourier coefficients of this elements, i.e., the following representation is true:

$$x(t) = \Phi(t)z. \tag{7}$$

As in [6], we conclude that the element $x(t)$ given by relations (7) is the required d_2-parameter family of solutions of the original boundary-value problem (1), (2). Therefore, the following statement is true.

Theorem 2. *Suppose that the generating boundary-value problem* (3) *is unsolvable. If conditions*

$$P_{\Lambda_r} P_{Q_{d_2}} \neq 0, \quad P_{B_0^*} G = 0,$$

are satisfied, then the boundary-value problem (1), (2) *has a d_2-parameter family of solutions in the form of series with singularity at the point $\varepsilon = 0$ convergent for sufficiently small fixed $\varepsilon \in (0, \varepsilon_*]$.*

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On Solvability of Two-Point Boundary Value Problems with Separating Boundary Conditions for Linear Ordinary Differential Equations and Totally Positive Kernels

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We consider two-point boundary value problems for higher order linear ordinary differential equations

$$\begin{cases} x^{(n)}(t)+p^{+}(t)x(t)-p^{-}(t)x(t)=f(t), & t\in[a,b],\\ x^{(i)}(0)=0, & i=0,\dots,m-1,\\ x^{(j)}(1)=0, & j=0,\dots,n-m-1, \end{cases} \tag{1}$$

where n, m are positive integer, $n>m$; $f\in\mathbf{L}[a,b]$, $p=p^{+}-p^{-}\in[a,b]$,

$$p^{+}(t)=\begin{cases} p(t) & \text{if } p(t)\geq 0,\\ 0 & \text{if } p(t)<0, \end{cases}\quad p^{-}(t)=\begin{cases} -p(t) & \text{if } p(t)<0,\\ 0 & \text{if } p(t)\geq 0, \end{cases}$$

$\mathbf{L}[a,b]$ is the space of Lebesgue integrable functions with the standard norm. Together with (1), we will consider some more general problems.

It is a rather common case, when problem (1) has a unique solution for all functions p^{+} (or for all functions p^{-}) with a fixed another function p^{-} (or p^{+}). So, our aim is to find some conditions to our boundary value problem (1) to be uniquely solvable for all integrable non-negative coefficients p^{+} (or for all nonnegative coefficients p^{-}).

In this report, we would like to remind about some classical results by F. R. Gantmacher, M. G. Krein, S. Karlin, A. Yu. Levin [1–5]. These results allow us to find required conditions in a very simple way. For higher-order equations, we don't know another proof of these conditions, for example, by means of mathematical analysis only.

A continuous function $G(\,\cdot\,,\,\cdot\,):[a,b]\times[a,b]\to R$ is called a totally positive kernel [3] if all determinants

$$\begin{vmatrix} G(t_1,t_1) & \dots & G(t_1,t_k)\\ \vdots & \ddots & \vdots\\ G(t_k,t_1) & \dots & G(t_k,t_k) \end{vmatrix}$$

are positive for all ordered sets of points $a<t_1<\dots<t_k<b$ for all integer positive numbers k.

It is very hard to check this property directly. Fortunately, Green functions of many boundary value problems for ordinary differential equations possess this property. Now we can formulate a well-known statement on the spectrum of integral operators with totally positive kernels.

Let $G(t,s)$ be a totally positive kernel, $\mathbf{C}[a,b]$ be the space of real continuous functions with the standard norm.

Consider the integral operator $G : \mathbf{C}[a,b] \to \mathbf{C}[a,b]$

$$(Gx)(t) = \int_a^b G(t,s)x(s)\,ds, \quad t \in [a,b]. \tag{2}$$

Theorem 1 (Sturm, Kellogg, Gantmacher, Krein, Karlin, Levin, Stepanov)**.** *The spectrum of the operators G is a subset of the set $[0,\infty)$.*

In the symmetric case, oscillating properties of the spectrum were known to Kellogg and Sturm. F. R. Gantmacher and M. G. Krein [1] showed that kernels could be non-symmetric and proved oscillation properties of the spectrum of many boundary value problems. Here we need only the positivity of the spectrum and we do not mention all remarkable oscillation properties. So, if the kernel $G(t,s)$ is totally positive, then the non-zero spectrum of operator (2) is positive. The next obvious step is only a more general formulation.

Let $G(t,s)$ be a totally positive kernel, $r \in \mathbf{L}[a,b]$, $r(t) \geq 0$, $t \in [a,b]$.

Consider the integral operator $G_r : \mathbf{C}[a,b] \to \mathbf{C}[a,b]$

$$(G_r x)(t) = \int_a^b G(t,s)r(s)x(s)\,ds, \quad t \in [a,b].$$

Theorem 2 (Sturm–Kellogg–Gantmacher–Krein)**.** *The spectrum of the operators G_r is a subset of the set $[0,\infty)$.*

Therefore, in this case all characteristic values λ of the equation

$$x(t) = \lambda \int_a^b G(t,s)r(s)x(s)\,ds, \quad t \in [a,b],$$

are positive.

Now consider two-point boundary value problems for linear higher order ordinary differential equations

$$\begin{cases} (Lx)(t) \equiv x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \cdots + p_n(t)x(t) = f(t), \\ \ell_i x \equiv \sum\limits_{k=0}^{k_i-1} \gamma_{ik} x^{(k)}(a) + x^{(k_i)}(a), \quad i = 1, \dots, m, \\ \ell_i x \equiv \sum\limits_{k=0}^{k_i-1} \gamma_{ik} x^{(k)}(b) + x^{(k_i)}(b), \quad i = m+1, \dots, n, \end{cases} \tag{3}$$

where $p_i \in \mathbf{L}[a,b]$; $f \in \mathbf{L}[a,b]$; n, m, $n > m$, are positive integers; $k_i \in \{0,1,\dots,n-1\}$, $i = 1,\dots,n$. Denote $\ell = \{\ell_1,\dots,\ell_n\}$.

Theorem 3. *If $r \in \mathbf{L}[a,b]$, $r(t) \geq 0$, $t \in [a,b]$, and the Green function $G(t,s)$ of this problem* (3) *is a totally positive kernel, then problem*

$$\begin{cases} (Lx)(t) + r(t)x(t) = f(t), \quad t \in [a,b], \\ \ell x = 0, \end{cases} \tag{4}$$

is uniquely solvable.

Proof. For $f \equiv 0$, (4) is equivalent to the integral equation

$$x(t) = \lambda \int\limits_a^b G(t,s) r(s) x(s)\, ds, \quad t \in [a,b], \tag{5}$$

where $\lambda = -1$. All eigenvalues of (5) are positive, therefore, problem (4) is uniquely solvable. $\square$

Together with the differential operator L, we consider operators L^+ and L^-:

$$\begin{aligned}
(L^+x)(t) &\equiv x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \cdots + p_n^+(t)x(t), \quad t \in [a,b],\\
(L^-x)(t) &\equiv x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \cdots - p_n^-(t)x(t), \quad t \in [a,b].
\end{aligned}$$

Theorem 4. *Let $G^+(t,s)$ be the Green function of the problem $L^+x = f$, $\ell x = 0$. If $-G^+(t,s)$ is a totally positive kernel, then the problem*

$$\begin{cases} (Lx)(t) = f(t), & \\ \ell_i x = 0, & i = 1, \dots, n, \end{cases} \tag{6}$$

is uniquely solvable for all non-negative functions $p_n^- \in \mathbf{L}[a,b]$.

Theorem 5. *Let $G^-(t,s)$ be the Green function of the problem $L^-x = f$, $\ell x = 0$. If $G^-(t,s)$ is a totally positive kernel, then problem* (6) *is uniquely solvable for all non-negative functions $p_n^+ \in \mathbf{L}[a,b]$.*

We say that the differential operator L (or the equation $Lx = 0$) is non-oscillating on the interval $[a,b]$ if every non-trivial solution has no more than $n-1$ zeros in the interval $[a,b]$ taking into account the multiplicity of the zeros. Hartman-Levin's criterion for non-oscillation can be found, for example, in [4].

Theorem 6 (Gantmacher–Krein, see [4, 5])**.** *Let $Lx = 0$ be non-oscillating, $G(t,s)$ the Green function of the problem*

$$\begin{cases} (Lx)(t) = f(t), & \\ x^{(i-1)}(a) = 0, & i = 1, \dots, m, \\ x^{(i-1)}(b) = 0, & i = 1, \dots, n-m. \end{cases}$$

Then $(-1)^{n-m}G(t,s)$ is a totally positive kernel.

Let the operator L be non-oscillating. Then L has the Polia–Mammana decomposition

$$Lx = r_0 \frac{d}{dt} r_1 \frac{d}{dt} \cdots r_{n-1} \frac{d}{dt} r_n,$$

where r_i, $i = 0, \dots, n$, are sufficiently smooth positive functions. Let $G(t,s)$ be the Green function of the uniquely solvable problem

$$\begin{cases} (Lx)(t) = f, & t \in [a,b], \\ \sum\limits_{k=1}^n \alpha_{ik}(D_{k-1})(a) = 0, & i = 1, \dots, m, \\ \sum\limits_{k=1}^n \beta_{ik}(D_{k-1})(b) = 0, & i = 1, \dots, n-m, \end{cases}$$

where $f \in \mathbf{L}[a,b]$; $D_0 x = x$, $D_k x = \frac{d}{dt}\,(r_{n-k+1} D_{k-1} x)$, $k = 1, \dots, n$; n, m, $n > m$, positive integers.

Theorem 7 (Kalafaty–Gantmacher–Krein, see [4, 5])**.** *If all m-th order minors of the matrix $||(-1)^k\alpha_{ik}||_{i=1,\dots,m}^{k=1,\dots,n}$ and all $(n-m)$-th order minors of $||\beta_{ik}||_{i=1,\dots,n-m}^{k=1,\dots,n}$ have the same sign, then $(-1)^{n-m}G(t,s)$ is a totally positive kernel.*

Theorem 8 (Levin–Stepanov, see [4,5])**.** *Let $G(t,s)$ be the Green function of the uniquely solvable problem*

$$\begin{cases} (Lx)(t)=f(t), & t\in[a,b],\\ \sum\limits_{k=0}^{k_i-1}\gamma_{ik}x^{(k)}(a)+x^{(k_i)}(a), & i=1,\dots,m,\\ \sum\limits_{k=0}^{k_i-1}\gamma_{ik}x^{(k)}(b)+x^{(k_i)}(b), & i=1,\dots,n-m, \end{cases}$$

$n_k=|\{i:\ k_i\le k, i=1,\dots,n\}|$, $h=2(b-a)$.

If $n_k>k$, $k=0,1,\dots,n-2$, and

$$\sum_{k=1}^{n}h^{k-1}\int\limits_a^b|p_k(t)|\,dt<\frac12\,,\quad \sum_{k=0}^{k_i-1}|\gamma_{ik}|h^{k_i-k}<\frac12\,,\ \ i=1,\dots n,$$

then $(-1)^{n-m}G(t,s)$ is a totally positive kernel.

Example 1. *The focal boundary value problem*

$$\begin{cases} x^{(n)}(t)+(-1)^{n-m}p(t)x(t)=f(t),\\ x^{(i)}(a)=0, & i=0,\dots,m-1,\\ x^{(i)}(b)=0, & i=m,\dots,n-1, \end{cases}$$

is uniquely solvable if $p(t)\ge0$, $t\in[a,b]$, $p\in\mathbf{L}[a,b]$.

Example 2. *Let $p^+\in\mathbf{L}[a,b]$, $p^+\ge0$, $t\in[a,b]$, and*

$$0\le p^-(t)\le\frac{24\cdot256}{27(b-a)^4}\,,\quad p^-(t)\not\equiv\frac{24\cdot256}{27(b-a)^4}\,. \tag{7}$$

Then the problem

$$\begin{cases} x^{(4)}(t)+p^+(t)x(t)-p^-(t)x(t)=f(t),\\ x(a)=0,\ \ \dot x(a)=0,\\ x(b)=0,\ \ \dot x(b)=0, \end{cases}$$

is uniquely solvable.

The constant in conditions (7) *are better than the constant $\frac{\pi^4}{(b-a)^4}$, which follows from Wirtinger's inequality.*

The conclusion: the classical results on totally positive kernels could be very useful for boundary value problems for ordinary differential equations.

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Asymptotic Properties of $P_\omega(Y_0, Y_1, \pm\infty)$-Solutions of Second Order Differential Equations with the Product of Different Classes of Nonlinearities

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We consider the following differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\varphi_1(y'). \tag{1}$$

In this equation $\alpha_0 \in \{-1;1\}$, functions $p : [a,\omega[\,\to\,]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$ and $\varphi_i : \Delta_{Y_i} \to\,]0,+\infty[$ $(i \in \{0,1\})$ are continuous, $Y_i \in \{0,\pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[$ or the interval $]Y_i, y_i^0]$. If $Y_i = +\infty$ $(Y_i = -\infty)$, we put $y_i^0 > 0$ $(y_i^0 < 0)$.

We also suppose that function φ_1 is a regularly varying as $y \to Y_1$ function of index σ_1 [10, p. 10-15], function φ_0 is twice continuously differentiable on Δ_{Y_0} and satisfies the next conditions

$$\varphi_0'(y) \ne 0 \ \text{ as } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \varphi_0(y) \in \{0,+\infty\}, \quad \lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1. \tag{2}$$

It follows from the above conditions (2) that the function φ_0 and its derivative of the first order are rapidly varying functions as the argument tends to Y_0 [10, p. 15]. Thus, the investigated differential equation contains the product of regularly and rapidly varying nonlinearities in its right-hand side.

The equations of the form (1) often appear in practice, for example, in the theory of burning, when we consider the electrostatic potential in a spherical volume of plasma of products of burning. First important results in this direction have been obtained in the works by V. M. Evtukhov for the equation of the investigated type in the case when $\varphi_0(y) = |y|^\sigma$ and $\varphi_1(y') = |y'|^\lambda$.

For the equation of the form (1), both functions φ_0 and φ_1 of which are regularly varying functions of orders σ_0 and σ_1 correspondingly $(\sigma_0 + \sigma_1 \ne 1)$ as their arguments tend to zero or to infinity, the asymptotic behavior of some class of solutions have been investigated in the works by M. O. Belozerova [5].

During investigations of distribution of electrostatic potential in a cylindrical plasma volume of combustion products, differential equation of the investigated type arises, in which $\varphi_0(y) = \exp(\sigma y)$, $\varphi_1(y') = |y'|^\lambda$, $\alpha_0 \in \{-1,1\}$, $\sigma,\lambda \in \mathbb{R}$, $\sigma \ne 0$, function $p : [a,\omega[\,\to\,]0,+\infty[$ $(-\infty < a < \omega \le +\infty)$ is a continuously differentiable function. Under some restrictions on the function $p(t)$ certain results for the asymptotic behavior of all regular solutions of this equation have been obtained in works by V. M. Evtukhov and N. G. Dric (see [7], for example).

Equations, that contain in their right-hand side the product of functions $\varphi_0(y)$ and $\varphi_1(y')$, the first one of which is a rapidly varying function as $y \to Y_0$ $(y \in \Delta_{Y_0})$, and the second one is a regularly varying function as $y' \to Y_1$ $(y' \in \Delta_{Y_1})$, in general case have not been investigated before. Thus, equation (1) plays an important role in the development of a qualitative theory of differential equations.

The main aim of the article is the investigation of conditions for the existence of following class of solutions of equation (1).

Definition 1. A solution y of equation (1), defined on the interval $[t_0,\omega[\subset [a,\omega[$, is called $P_\omega(Y_0,Y_1,\lambda_0)$-solution ($-\infty\le\lambda_0\le+\infty$) if the following conditions take place

$$y^{(i)}:[t_0,\omega[\to\Delta_{Y_i},\quad \lim_{t\uparrow\omega}y^{(i)}(t)=Y_i\ \ (i=0,1),\quad \lim_{t\uparrow\omega}\frac{(y'(t))^2}{y''(t)y(t)}=\lambda_0.$$

This class of solutions was defined in the work of V. M. Evtukhov [3] for the n-th order differential equations of Emden–Fowler type and was concretized for the second-order equation. Due to the asymptotic properties of functions in the class of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions [4], every such solution belongs to one of four non-intersecting sets according to the value of $\lambda_0:\lambda_0\in\mathbb{R}\setminus\{0,1\}$, $\lambda_0=0$, $\lambda_0=1$, $\lambda_0=\pm\infty$. In this article we consider the case $\lambda_0=\pm\infty$ of such solutions, every $P_\omega(Y_0,Y_1,\pm\infty)$-solution and its derivative satisfy the following limit relations

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)y'(t)}{y(t)}=1,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)y''(t)}{y'(t)}=0. \tag{3}$$

This class of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions for equations of the form (1) is one of the most difficult to study due to the fact that the second-order derivative is not explicitly expressed through the first-order derivative. From (3) it means that the derivative of the first order of each such solution is a slowly varying function as $t\uparrow\omega$.

From conditions (2) it also follows that the function φ_0 and its first-order derivative belong to the class $\Gamma_{Y_0}(Z_0)$, that was introduced in the works of V. M. Evtukhov and A. G. Chernikova [6] as a generalization of the class Γ (L. Khan, see, for example, [1, p. 75]). The properties of the class $\Gamma_{Y_0}(Z_0)$ were used to get our results.

To formulate the main results, we introduce the following definitions.

Definition 2. Let $Y\in\{0,\infty\}$, Δ_Y is some one-sided neighborhood of Y. Continuous-differentiable function $L:\Delta_Y\to]0,+\infty[$ is called [9, p. 2-3] a normalized slowly varying function as $z\to Y$ $(z\in\Delta_Y)$ if the next statement is valid

$$\lim_{\substack{y\to Y\\ y\in\Delta_Y}}\frac{yL'(y)}{L(y)}=0.$$

Definition 3. We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0,+\infty[$ satisfies the condition S as $z\to Y$, if for any continuous differentiable normalized slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $L:\Delta_{Y_i}\to]0,+\infty[$ the next relation is valid

$$\theta(zL(z))=\theta(z)(1+o(1))\ \text{ as }\ z\to Y\ \ (z\in\Delta_Y).$$

Condition S is satisfied, for example, for such functions as $\ln|y|$, $|\ln|y||^\mu$ $(\mu\in\mathbb{R})$, $\ln\ln|y|$.

The following theorem is obtained in our previous work [2] and contains a necessary conditions for the existence of the $P_\omega(Y_0,Y_1,\pm\infty)$-solution of equation (1).

Theorem 1 ([2])**.** *Let for equation* (1) $\sigma_1\neq1$*, the function* $\varphi_1(y')|y'|^{-\sigma_1}$ *satisfy the condition* S *as* $y'\to Y_1$ $(y'\in\Delta_{Y_1})$*. Then each* $P_\omega(Y_0,Y_1,\pm\infty)$*-solution of the differential equation* (1) *can be represented as*

$$y(t)=\pi_\omega(t)L(t),$$

where $L:[t_0,\omega[\to\mathbb{R}$ *is twice continuously differentiable on* Δ_{Y_0} *and satisfies the next conditions*

$$y_0^0\pi_\omega(t)L(t)>0,\quad L'(t)\neq0\ \text{ as }\ t\in[t_1,\omega[\ \ (t_0\le t_1<\omega),$$

$$\lim_{t\uparrow\omega}L(t)\in\{0;\pm\infty\},\quad \lim_{t\uparrow\omega}\pi_\omega(t)L(t)=Y_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)L'(t)}{L(t)}=0. \tag{4}$$

Thus, in the case of the existence of a finite or infinite limit

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)L''(t)}{L'(t)}, \tag{5}$$

the following relations take place

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)L''(t)}{L'(t)}=-1,\quad \alpha_0L'(t)>0\ \text{ as }\ t\in[t_1,\omega[\ \ (t_0\le t_1<\omega), \tag{6}$$
$$p(t)=\frac{\alpha_0L'(t)}{\varphi_1(L(t))\varphi_0(\pi_\omega(t)L(t))}\,[1+o(1)]\ \text{ as }\ t\uparrow\omega.$$

Let us introduce the following definition.

Definition 4. We say that the condition N is satisfied for equation (1) if for some continuously differentiable function $L(t):[t_0,\omega[\,\to\mathbb{R}$ ($t_0\in[a,\omega[$), which satisfies conditions (4)–(6), the following representation takes place

$$p(t)=\frac{\alpha_0L'(t)}{\varphi_1(L(t))\varphi_0(\pi_\omega(t)L(t))}\,[1+r(t)],$$

where $r(t):[t_0,\omega[\,\to\,]-1,+\infty[$ is a continuous function that tends to zero as $t\uparrow\omega$.

For equation (1), in previous works [2] the necessary and sufficient conditions for the existence of the investigated class of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions were established in case of the existence of some infinite limit. In this work we establish sufficient conditions for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of equation (1) in case this limit equals nonzero real number. We also have found the asymptotic representations of such solutions and its first order derivatives as $t\uparrow\omega$ and indicated the number of such solutions.

To formulate the sufficient conditions for the existence of the $P_\omega(Y_0,Y_1,\pm\infty)$-solution of equation (1), let us introduce some notations:

$$\mu_0=\operatorname{sign}\varphi_0'(y),\quad \theta_1(y')=\varphi_1(y')|y'|^{-\sigma_1},$$
$$H(t)=\frac{L^2(t)\varphi_0'(\pi_\omega(t)L(t))}{L'(t)\varphi_0(\pi_\omega(t)L(t))},\quad q_1(t)=\frac{\big(\frac{\varphi_0'(y)}{\varphi_0(y)}\big)'}{\big(\frac{\varphi_0'(y)}{\varphi_0(y)}\big)^2}\Bigg|_{y=\pi_\omega(t)L(t)},$$
$$e_1(t)=1+\frac{\pi_\omega(t)L'(t)}{L(t)}\,,\quad e_2(t)=2+\frac{\pi_\omega(t)L''(t)}{L'(t)}\,.$$

For these functions the following statements are fulfilled:

1)
$$\lim_{t\uparrow\omega}e_1(t)=\lim_{t\uparrow\omega}e_2(t)=1,\quad \lim_{t\uparrow\omega}H(t)=\pm\infty,\quad \lim_{t\uparrow\omega}q_1(t)=0,$$

2) If the limit
$$\lim_{t\uparrow\omega}\frac{L(t)}{L'(t)}\cdot\frac{H'(t)}{|H(t)|^{\frac32}}$$
exists, then
$$\lim_{t\uparrow\omega}\frac{L(t)}{L'(t)}\cdot\frac{H'(t)}{|H(t)|^{\frac32}}=0. \tag{7}$$

The sufficient conditions for the existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions of equation (1) in case

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)L'(t)}{L(t)}\,|H(t)|^{\frac{1}{2}} = \pm\infty,$$

were found in [2].

In this work we suppose that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)L'(t)}{L(t)}\,|H(t)|^{\frac{1}{2}} = \gamma, \quad 0 < |\gamma| < +\infty. \tag{8}$$

The sufficient conditions for this case are formulated in the following theorem.

Theorem 2. *Let for equation* (1) $\sigma_1 \neq 1$, *the function* $\varphi_1(y')|y'|^{-\sigma_1}$ *satisfy the condition* S *as* $y' \to Y_1$ $(y' \in \Delta_{Y_1})$, *the conditions* N, (7) *and* (8) *hold. Then*

- *in case* $\alpha_0\mu_0 > 0$, *the differential equation* (1) *has a one-parametric family of* $P_\omega(Y_0, Y_1, \pm\infty)$-*solutions;*

- *in case* $\alpha_0\mu_0 < 0$ *and* $y_0^0\alpha_0\gamma\pi_\omega(t) < 0$, *the differential equation* (1) *has a two-parametric family of* $P_\omega(Y_0, Y_1, \pm\infty)$-*solutions;*

- *in case* $\alpha_0\mu_0 < 0$ *and* $y_0^0\alpha_0\gamma\pi_\omega(t) > 0$, *the differential equation* (1) *has at least one of* $P_\omega(Y_0, Y_1, \pm\infty)$-*solutions.*

For each of such solutions the following asymptotic representations take place as $t \uparrow \omega$,

$$\begin{gathered} y(t) = \pi_\omega(t)\cdot L(t) + \frac{\varphi_0(\pi_\omega(t)L(t))}{\varphi_0'(\pi_\omega(t)L(t))}\cdot o(1), \\ y'(t) = \big[L(t) + \pi_\omega(t)\cdot L'(t)\big]\cdot\big[1 + |H(t)|^{-\frac{1}{2}}\cdot o(1)\big]. \end{gathered}$$

For the equation under the investigation the question of the active existence of $P_\omega(Y_0, Y_1, \pm\infty)$-solutions, that have the obtained asymptotic representations, has been reduced to the question of the existence of infinitely small as arguments tend to ω solutions of the corresponding, equivalent to the investigated equation, systems of non-autonomous quasi-linear differential equations that admit applications of the known results from the works of V. M. Evtukhov and A. M. Samoilenko [8].

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Boundary Value Problems for Systems of Nonsingular Integral-Differential Equations of Fredholm Type with Degenerate Kernel

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We investigate the problem of finding solutions [1]

$$y(t)\in\mathbb{D}^2[a;b],\ \ y'(t)\in\mathbb{L}^2[a;b]$$

of the linear Noetherian ($n\neq v$) boundary value problem for a system of linear integral-differential equations of Fredholm type with degenerate kernel

$$A(t)y'(t)=B(t)y(t)+\Phi(t)\int\limits_a^b F(y(s),y'(s),s)\,ds+f(t),\ \ \ell y(\,\cdot\,)=\alpha,\ \ \alpha\in\mathbb{R}^p. \tag{1}$$

We seek a solution of the Noetherian boundary value problem (1) in a small neighborhood of the solution

$$y_0(t)\in\mathbb{D}^2[a;b],\ \ y_0'(t)\in\mathbb{L}^2[a;b]$$

of the generating problem

$$A(t)y_0'(t)=B(t)y_0(t)+f(t),\ \ \ell y_0(\,\cdot\,)=\alpha. \tag{2}$$

Here

$$A(t),B(t)\in\mathbb{L}^2_{m\times n}[a;b]:=\mathbb{L}^2[a;b]\otimes\mathbb{R}^{m\times n},\ \ \Phi(t)\in\mathbb{L}^2_{m\times q}[a;b],\ \ f(t)\in\mathbb{L}^2[a;b].$$

We assume that the matrix $A(t)$ is, generally speaking, rectangular: $m\neq n$. It can be square, but singular. Assume that the function $F(y(t),y'(t),t)$ is linear with respect to unknown $y(t)$ in a small neighborhood of the generating solutions and with respect to the derivative $y'(t)$ in a small neighborhood of the function $y_0'(t)$. In addition, we assume that the function $F(y(t),y'(t),t)$ is continuous in the independent variable t on the segment $[a,b]$;

$$\ell y(\,\cdot\,):\mathbb{D}^2[a;b]\to\mathbb{R}^p$$

is a linear bounded vector functional defined on a space $\mathbb{D}^2[a;b]$. The problem of finding solutions of the boundary value problem (1) in case $A(t)=I_n$ was solved by A. M. Samoilenko and A. A. Boichuk [7]. Thus, the boundary value problem (1) is a generalization of the problem solved by A. M. Samoilenko and A. A. Boichuk.

We investigate the problem of finding solutions of the linear Noetherian boundary value problem (2) in the paper [9]. Under the condition

$$P_{A^*}(t)=0,\ \ \operatorname{rank}A(t):=\sigma_0=m\leq n \tag{3}$$

we arrive at the problem of construction of solutions of the linear differential-algebraic system [9]

$$z' = A^+(t)B(t)z + \mathfrak{F}_0(t,\nu_0(t)); \tag{4}$$

here,

$$\mathfrak{F}_0(t,\nu_0(t)) := A^+(t)f(t) + P_{A_{\rho_0}}(t)\nu_0(t),$$

$A^+(t)$ is a pseudoinverse (by Moore–Penrose) matrix [1], $\nu_0(t) \in \mathbb{L}^2[a;b]$ is an arbitrary vector function. In addition, $P_{A^*(t)}$ is a matrix-orthoprojector [1]:

$$P_{A^*}(t) : \mathbb{R}^m \to \mathbb{N}(A^*(t)),$$

$P_{A_{\rho_0}}(t)$ is an $(n \times \rho_0)$-matrix composed of ρ_0 linearly independent columns of the $(n \times n)$-matrix-orthoprojector:

$$P_A(t) : \mathbb{R}^n \to \mathbb{N}(A(t)).$$

By analogy with the classification of pulse boundary-value problems [1–3,8] we say in the (3), that, system of linear integral-differential equations is nonsingular. Denote by $X(t)$ normal fundamental matrix

$$X'(t) = A^+(t)B(t)X(t), \quad X(a) = I_n.$$

Substituting the general solution of the system of linear integral-differential equations (3) into the boundary condition (1), we arrive at the linear algebraic equation

$$Q\,c = \ell K[f(s)](\,\cdot\,). \tag{5}$$

In the critical case

$$P_{Q^*} \neq 0, \ \ Q := \ell X(\,\cdot\,) \in \mathbb{R}^{p\times n},$$

equation (5) is solvable iff

$$P_{Q_d^*}\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} = 0. \tag{6}$$

Here, $P_{Q_d^*}$ is an $(d \times p)$-matrix composed of d linearly independent rows of the $(p \times p)$-matrix-orthoprojector:

$$P_{Q^*} : \mathbb{R}^p \to \mathbb{N}(Q^*).$$

Thus, the following lemma is proved [9].

Lemma 1. *In the critical case $P_{Q^*} \neq 0$, the nonsingular differential-algebraic boundary value problem* (2) *is solvable iff* (6) *holds. In the critical case, the nonsingular differential-algebraic boundary value problem* (2) *has a solution of the form*

$$y_0(t,c_r) = X_r(t)c_r + G[f(s);\alpha](t), \quad X_r(t) := X(t)P_{Q_r}, \ \ c_r \in \mathbb{R}^r,$$

which depends on the arbitrary vector-function $\nu_0(t) \in \mathbb{L}^2[a;b]$. Here, P_{Q_r} is an $(p\times r)$-matrix composed of r linearly independent columns of the $(p\times p)$-matrix-orthoprojector: $P_Q : \mathbb{R}^p \to \mathbb{N}(Q)$;

$$G[f(s);\alpha](t) := X(t)Q^+\big\{\alpha - \ell K[f(s)](\,\cdot\,)\big\} + K[f(s)](t)$$

is the generalized Green operator of the linear integral-differential problem (1)*;*

$$K\big[f(s)\big](t) := X(t)\int\limits_a^t X^{-1}(s)\mathfrak{F}_0(s,\nu_0(s))\,ds$$

is the generalized Green operator of the Cauchy problem for the integral-differential system (3)*.*

In problem (1) we perform the substitution $y(t)=y_0(t,c_r)+x(t)$. For

$$x(t)\in\mathbb{D}^2[a;b],\ \ x'(t)\in\mathbb{L}^2[a;b],\ \ x(t,u,v)=X_0(t)u+\Psi(t)v$$

we obtain the problem

$$A(t)x'(t)=B(t)x(t)+\Phi(t)\int\limits_a^b F(y(s),y'(s),s)\,ds,\ \ \ell x(\,\cdot\,)=0.$$

Here,

$$v:=\int\limits_a^b F(y(s),y'(s),s)\,ds\in\mathbb{R}^q,\ \ u\in\mathbb{R}^n,\ \ \Psi(t):=K[\Phi(s)](t)\in\mathbb{D}^2_{n\times q}[a;b].$$

Denote the matrix

$$\check{Q}:=[Q;R]\in\mathbb{R}^{p\times(q+n)},\ \ R:=\ell\,\Psi(\,\cdot\,)\in\mathbb{R}^{p\times q}$$

and $P_\rho\in\mathbb{R}^{(q+n)\times\rho}$ composed of ρ linearly independent columns of the matrix-orthoprojector $P_{\check{Q}}$:

$$P_{\check{Q}}:\mathbb{R}^{q+n}\to\mathbb{N}(\check{Q}).$$

Substituting the general solution of the system of the linear integral-differential system (1) into the boundary condition (1), we arrive at the linear algebraic equation

$$Q\,v+R\,u=0,\ \ R:=\ell\,\Psi(\,\cdot\,)\in\mathbb{R}^{p\times q}.$$

Using the continuous differentiability of the function $F(y(t),y'(t),t)$ with respect to unknown $y(t)$ in a small neighborhood of the generating solutions and with respect to the derivative $y'(t)$ in a small neighborhood of the function $y_0'(t)$, we expand this function

$$F(y(t),y'(t),t)=A_1(t)y(t)+A_2(t)y'(t),\ \ A_1(t):=F'_y(y(t),y'(t),t),\ \ A_2(t):=F'_{y'}(y(t),y'(t),t).$$

Applying Lemma 1 to the boundary value problem (1), we obtain equation

$$\mathcal{B}_0\,c_\rho+\psi(c_r)=0,\quad \psi(c_r):=-\int\limits_a^b\big[A_1(t)y_0(t,c_r)+A_2(t)y_0'(t,c_r)\big]\,dt,\tag{7}$$

where

$$\mathcal{B}_0:=\int\limits_a^b\Big\{A_1(t)[X(t)P_1+\Psi(t)P_2]+A_2(t)\big[X'(t)P_1+\Psi'(t)P_2\big]\Big\}\,dt-P_2,$$

In the critical case $P_{\mathcal{B}_0^*}\neq0$, equation (6) is solvable iff

$$P_{\mathcal{B}_0^*}\psi(c_r)=0.\tag{8}$$

Here,

$$P_{\mathcal{B}_0^*}:\mathbb{R}^q\to\mathbb{N}(\mathcal{B}_0^*),\quad P_{\mathcal{B}_0}:\mathbb{R}^\rho\to\mathbb{N}(\mathcal{B}_0)$$

is matrices-orthoprojectors. Under condition (8) and only under it nonsingular system of linear integral-differential equations (1) has a solution of the form

$$y(t,c_\mu)=Y_\mu(t)c_\mu+G[\Phi(s);\nu_0(s)](t),\ \ c_\mu\in\mathbb{R}^\mu,$$

which depends on the arbitrary vector-function $\nu_0(t) \in \mathbb{L}^2[a;b]$. Here,

$$G[\Phi(s);\nu_0(s)](t) := G[f(s);\alpha](t) - \mathcal{B}_0^+ \int\limits_a^b \Big[A_1(t)G[f(s);\alpha](t) + A_2(t)G'[f(s);\alpha](t)\Big]\,dt$$

is the generalized Green operator of the linear integral-differential problem (1), where $Y_\mu(t)$ is an $(n \times \mu)$-matrix composed of μ linearly independent columns of the matrix:

$$\Big[X_0(t) - \big[X_0(t)P_1 + \Psi(t)P_2\big]\mathcal{B}_0^+\big[A_1(t)X_r(t); A_2(t)X_r'(t)\big]\,dt; \big[X(t)P_1 + \Psi(t)P_2\big]P_{\mathcal{B}_0}\Big].$$

Thus, the following theorem is proved.

Theorem 1. *In the critical case, under condition* (6) *the nonsingular integral-differential boundary value problem* (3) *has a solution of the form*

$$y_0(t,c_r) = X_r(t)c_r + G[f(s);\alpha](t), \quad X_r(t) := X(t)P_{Q_r}, \ \ c_r \in \mathbb{R}^r,$$

which depends on the arbitrary vector-function $\nu_0(t) \in \mathbb{L}^2[a;b]$. *Under condition* (8) *and only under it the general solution of the nonsingular integral-differential boundary value problem* (1)

$$y(t,c_\mu) = Y_\mu(t)c_\mu + G[\Phi(s);\nu_0(s)](t), \ \ c_\mu \in \mathbb{R}^\mu$$

is determined by the generalized Green operator of the nonsingular integral-differential boundary value problem (1).

The proposed scheme of studies of the nonsingular integral-differential boundary value problem (1) can be transferred analogously to [1, 5, 6] onto nonlinear nonsingular integral-differential boundary value problem. On the other hand, in the case of nonsolvability, the nonsingular integral-differential boundary value problems can be regularized analogously [4, 10].

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A Condition for the Solvability of the Control Problem of Asynchronous Spectrum of Linear Almost Periodic Systems with the Diagonal Averaging of Coefficient Matrix

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A large number of works are devoted to the study of various questions of control theory for ordinary periodic differential systems (see, for example, [6, 11, 12] and others). For almost periodic control systems, such studies are significantly complicated. In this direction, we can note the results of [5, 9, 10], a characteristic feature of which is the consideration of the so-called regular case, when a priori it is assumed that the frequencies of the system itself and its solutions coincide.

At the same time, as shown by J. Kurzweil and O. Vejvoda [7], the system of ordinary differential almost periodic equations can admit such solutions that the intersection of the frequency modules of the solution and the system is trivial. This result allows us to assume that there exist systems with a very difference spectrum of frequencies, including asynchronous.

In [1], the control problem of the asynchronous spectrum for periodic systems was first formulated. A series of conditions for its solvability are given in the monograph [2, Ch. III]. Similar questions for quasiperiodic systems were studied in [3]. The control problem of the asynchronous spectrum of linear almost periodic systems was formulated in [4] and the case of trivial mean value of the coefficient matrix is considered.

Now we study the solvability of the control problem of the asynchronous spectrum of linear almost periodic systems for which the mean value of the coefficient matrix is diagonal.

Let $f(t)$ be a real almost periodic (Borh) function [2]. The mean value of an almost periodic function $f(t)$ is determined by the equality

$$\widehat{f}=\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T f(t)\,dt.$$

The modulus (frequency modulus) $\mathrm{Mod}(f)$ of an almost periodic function $f(t)$ is the smallest additive group of real numbers that contains all the Fourier exponents (frequencies) of this function. Let be $g(t,x)$ a vector-function that is almost periodic in uniformly relative to some compact set. J. Kurzweil and O Vejvoda proved that the system of ordinary differential equations

$$\dot{x}=g(t,x)$$

can have an almost periodic solution $x(t)$ such that the intersection of the frequency modules of the solution and the right-hand side is trivial, i.e.

$$\mathrm{Mod}(x)\cap\mathrm{Mod}(g)=\{0\}.$$

In what follows, such almost periodic solutions will be called strongly irregular.

Consider the linear non-stationary control system

$$\dot{x} = A(t)x + Bu, \ \ t \in \mathbb{R}, \ \ x \in \mathbb{R}^n, \ \ n \geq 2, \tag{1}$$

where x is the phase vector of the system, u is the input, B is the constant $n \times n$-matrix under control, $A(t)$ is a continuous almost periodic matrix with a modulus of frequencies $\operatorname{Mod}(A)$. Suppose that the control is specified in the form of a feedback linear in the phase variables

$$u = U(t)x$$

with a continuous almost periodic $n \times n$-matrix $U(t)$ (feedback coefficient), the frequency modulus of which is contained in the frequency modulus of the coefficient matrix, i.e.,

$$\operatorname{Mod}(U) \subseteq \operatorname{Mod}(A).$$

It is required to obtain conditions on the right-hand side of system (1) such that for any choice of the feedback coefficient from the indicated admissible set, the closed-loop system

$$\dot{x} = (A(t) + BU(t))x,$$

does not have a strongly irregular almost periodic solution, the frequency spectrum of which contains a given subset (target set). In other words, for system (1) it is necessary to find the conditions for the unsolvability of the problem of control of the asynchronous spectrum.

We suppose that the coefficient matrix has a diagonal average value, i.e.,

$$\widehat{A} = \operatorname{diag}(\widehat{a}_{11}, \dots, \widehat{a}_{nn}), \quad \widehat{a}_{11}^2 + \cdots \widehat{a}_{nn}^2 \neq 0. \tag{2}$$

Consider the case when the matrix under control is singular, i.e.

$$\operatorname{rank} B = r \ \ (1 \leq r < n), \tag{3}$$

moreover, its first rows are zero. Let us denote by $B_{r,n}$ the matrix composed of the remaining rows of the matrix B. The rank of the matrix $B_{r,n}$ is also equal to r. Taking into account the representation (3) of the matrix B, the matrix of coefficients $A(t)$ is divided into four blocks of the corresponding dimensions (indicated by the subscripts):

$$A(t) = \begin{pmatrix} A_{d,d}^{(11)}(t) & A_{d,r}^{(12)}(t) \\ A_{r,d}^{(21)}(t) & A_{r,r}^{(22)}(t) \end{pmatrix}.$$

Taking into account condition (2), we write the average value of the coefficient matrix in the form

$$\widehat{A} = \begin{pmatrix} \widehat{A}_{d,d}^{(11)} & 0 \\ 0 & \widehat{A}_{r,r}^{(22)} \end{pmatrix},$$

where $\widehat{A}_{d,d}^{(11)} = \operatorname{diag}(\widehat{a}_{11}, \dots, \widehat{a}_{dd})$, $\widehat{A}_{r,r}^{(22)} = \operatorname{diag}(\widehat{a}_{d+1\,d+1}, \dots, \widehat{a}_{nn})$. Then the oscillating part of the coefficient matrix is also represented in the following block form:

$$\widetilde{A}(t) = A(t) - \widehat{A} = \begin{pmatrix} \widetilde{A}_{d,d}^{(11)}(t) & A_{d,r}^{(12)}(t) \\ A_{r,d}^{(21)}(t) & \tilde{A}_{r,r}^{(22)} \end{pmatrix}.$$

Denote by q the column rank of a rectangular $d \times n$-matrix $(\widetilde{A}_{d,d}^{(11)} \ A_{d,r}^{(12)})$.

The following theorem holds.

Theorem. *Let conditions* (3) *and the equality* $q = n$ *hold. Then the problem of control of the asynchronous spectrum of system* (1) *has no solution.*

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Asymptotic Properties of Solutions to Half-Linear Differential Equation with Negative Potential

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1 Introduction

We present a complete overview on the qualitative behavior of solutions of the half-linear equation

$$(a(t)|x'|^{\alpha}\operatorname{sgn} x')' - b(t)|x|^{\alpha}\operatorname{sgn} x = 0, \tag{1.1}$$

where α is a positive constant and the functions a, b are continuous and positive for $t \ge t_0 \ge 0$.

Equation (1.1) comes out in studying radial solutions of equations with p-Laplacian operator and have been widely investigated in the literature.

The study on the qualitative behavior of solutions of (1.1), especially as concerns the classification of solutions, the existence of monotone bounded or unbounded solutions, the growth at infinity or the decay at zero of solutions, has a long history. Many of the results obtained in these fields have been obtained for more general equations and it would be impossible to mention all of them. As regards in particular the half-linear case, we recall the pioneering works of Elbert and Mirzov [13, 25] and we refer the reader for more details to the monographs [12, 26] and references therein.

Interesting contributions are due to the Georgian and Russian mathematical school. Almost all of these papers concern very general differential equations, which include, in particular, the Emden–Fowler equation or the Thomas-Fermi equation, see [4, 6, 7, 16, 18, 22, 28]. Recently, other developments are given by the Japanese mathematical school, see [14,15,19,20,27,30] under different point of view.

Our aim here is to present a complete overview, *quadro completo*, to the asymptotic behavior of solutions of (1.1). This result is a generalization of the one corresponding for the linear equation with Sturm–Liouville differential operator, see, e.g. [5]. In particular, we show that when the functions a, b have, roughly speaking, a power behavior near infinity, then the complete overview is the same as for the linear equation. Our approach follows that one in [1–3], even if here the results are more complete and the method is slightly different.

2 Basic properties

Following the linear case, we introduce a classification of solutions of (1.1), which is based on the one in [1–3], with minor modifications. We note that a slightly different classification of solutions of an equation, which includes (1.1), has been used in [27] under the additional assumption $a^{-1/\alpha} \notin L^1[t_0,\infty)$ and in [30] in the opposite situation $a^{-1/\alpha} \in L^1[t_0,\infty)$. Passing from the linear case to the half-linear one, it is well-known that several basic differences arise, see, e.g., [12, Section 1.3]. In particular, the set of solutions of (1.1), $\alpha \neq 1$, is not a linear space, the Jacobi–Liouville identity for the Wronskian and the variation of constant principle fail to hold for (1.1) with $\alpha \neq 1$. Independently of this fact, there are also a lot of similarities in asymptotic behavior of solutions.

Recall that any nontrivial solution x of (1.1) is defined on the whole interval $[t_0,\infty)$ and satisfies $\sup\limits_{t\in[\tau,\infty)} |x(t)| > 0$ for any $\tau \geq t_0$. Moreover, the Cauchy problem for (1.1) is uniquely solvable for any couple of initial data. In other words, given $T \geq t_0$ and $x_0, x_1 \in \mathbb{R}$, there exists a unique solution x of (1.1) satisfying $x(T) = x_0$, $x'(T) = x_1$ and x is defined on the whole interval $[t_0,\infty)$. Consequently, $x \equiv 0$ if and only if $x_0 = x_1 = 0$. Further, equation (1.1) is disconjugate on $[t_0,\infty)$, that is any nontrivial solution of (1.1) has at most one zero on $[t_0,\infty)$. Hence, (1.1) is nonoscillatory. The following holds.

Theorem 2.1. *The set of nontrivial solutions of* (1.1) *may be divided into two classes*

$$\begin{aligned} \mathbb{M}^+ &= \big\{x \text{ solution of } (1.1): \ \exists\, t_x \geq t_0: \ x(t)x'(t) > 0 \text{ for } t > t_x\big\},\\ \mathbb{M}^- &= \big\{x \text{ solution of } (1.1): \ x(t)x'(t) < 0 \text{ for } t > t_0\big\}, \end{aligned}$$

and both classes are nonempty. In particular, solutions x of (1.1)*, satisfying either $x(T) = 0$, $x'(T) > 0$ or $x(T) > 0$, $x'(T) = 0$ at some $T \geq t_0$, are positive increasing on (T,∞) and belong to the class $\mathbb{M}^+$. Further, if a solution of* (1.1) *in the class $\mathbb{M}^+$ is bounded, then every solution in the class $\mathbb{M}^+$ is bounded, too.*

The proof of Theorem 2.1 follows an idea used by Mambriani to solve the well-known Thomas-Fermi problem, see [29, Chapter XII, Section 5.]. An alternative proof can be found in [7].

The asymptotic behavior of solutions of (1.1) depends on the four integrals

$$J_1 = \int_{t_0}^{\infty} a^{-1/\alpha}(t)\Big(\int_{t_0}^{t} b(r)\,dr\Big)^{1/\alpha} dt, \quad J_2 = \int_{t_0}^{\infty} a^{-1/\alpha}(t)\Big(\int_{t}^{\infty} b(r)\,dr\Big)^{1/\alpha} dt,$$

$$Y_1 = \int_{t_0}^{\infty} b(t)\Big(\int_{t}^{\infty} a^{-1/\alpha}(r)\,dr\Big)^{\alpha} dt, \quad Y_2 = \int_{t_0}^{\infty} b(t)\Big(\int_{t_0}^{t} a^{-1/\alpha}(r)\,dr\Big)^{\alpha} dt.$$

A complete classification of solutions (1.1) require a preliminary analysis of mutual behavior of these integrals. Using some integral inequalities, we get the following.

Lemma 2.1 ([11]). *If $\alpha \geq 1$, then*

$$Y_2 = \infty \implies J_2 = \infty, \quad Y_1 = \infty \implies J_1 = \infty.$$

If $0 < \alpha \leq 1$, then

$$J_2 = \infty \implies Y_2 = \infty, \quad J_1 = \infty \implies Y_1 = \infty.$$

Lemma 2.1 can be viewed as an extension of the Fubini theorem. Indeed, when $\alpha = 1$, we have

$$J_1 = Y_1, \quad J_2 = Y_2. \tag{2.1}$$

By virtue of Lemma 2.1, the possible cases concerning the convergence of integrals $J_i, Y_i, i = 1,2$, are the following eight:

(C_1): $J_1 = \infty$, $J_2 = \infty$, $Y_1 = \infty$, $Y_2 = \infty$, $\alpha > 0$;

(C_2): $J_1 = \infty$, $J_2 < \infty$, $Y_1 = \infty$, $Y_2 < \infty$, $\alpha > 0$;

(C_3): $J_1 < \infty$, $J_2 = \infty$, $Y_1 < \infty$, $Y_2 = \infty$, $\alpha > 0$;

(C_4): $J_1 < \infty$, $J_2 < \infty$, $Y_1 < \infty$, $Y_2 < \infty$, $\alpha > 0$.

(C_5): $J_1 = \infty$, $J_2 = \infty$, $Y_1 = \infty$, $Y_2 < \infty$, $\alpha > 1$;

(C_6): $J_1 = \infty$, $J_2 = \infty$, $Y_1 < \infty$, $Y_2 = \infty$, $\alpha > 1$;

(C_7): $J_1 = \infty$, $J_2 < \infty$, $Y_1 = \infty$, $Y_2 = \infty$, $0 < \alpha < 1$;

(C_8): $J_1 < \infty$, $J_2 = \infty$, $Y_1 = \infty$, $Y_2 = \infty$, $0 < \alpha < 1$.

All cases (C_n), $n = 1, \dots, 8$, may occur, as examples below. Cases (C_1)–(C_4), may occur for any $\alpha > 0$. Cases (C_5) and (C_6) may occur only when $\alpha > 1$, and cases (C_7), (C_8) only when $0 < \alpha < 1$. Thus, cases (C_5)–(C_8), do not occur in the linear case and so, roughly speaking, they are typical for the half-linear case. When $\alpha = 1$, that is for the linear equation, the possible cases are only the four cases (C_1)–(C_4). Moreover, in view of (2.1), for the linear equation the integrals Y_1 and Y_2 do not play any role.

3 A complete overview

A precise and complete classification of solutions x of (1.1) may be done by considering also the asymptotic behavior of the quasiderivative $x^{[1]}$, that is the function $x^{[1]}(t) = a(t)|x'(t)|^{\alpha} \operatorname{sgn} x'(t)$.

Any solution of (1.1) in the class $\mathbb{M}^+$ belongs to one of the following four subclasses:

$$\begin{aligned}
\mathbb{M}^+_{\infty,\infty} &= \Big\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |x^{[1]}(t)| = \infty\Big\},\\
\mathbb{M}^+_{\infty,\ell} &= \Big\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |x^{[1]}(t)| < \infty\Big\},\\
\mathbb{M}^+_{\ell,\infty} &= \Big\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| < \infty, \ \lim_{t\to\infty} |x^{[1]}(t)| = \infty\Big\},\\
\mathbb{M}^+_{\ell,\ell} &= \Big\{x \in \mathbb{M}^+ : \lim_{t\to\infty} |x(t)| < \infty, \ \lim_{t\to\infty} |x^{[1]}(t)| < \infty\Big\}.
\end{aligned}$$

Similarly, any solution of (1.1) in the class $\mathbb{M}^-$ belongs to one of the following four subclasses:

$$\begin{aligned}
\mathbb{M}^-_{\ell,\ell} &= \Big\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) \neq 0, \ \lim_{t\to\infty} x^{[1]}(t) \neq 0\Big\},\\
\mathbb{M}^-_{\ell,0} &= \Big\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) \neq 0, \ \lim_{t\to\infty} x^{[1]}(t) = 0\Big\},\\
\mathbb{M}^-_{0,\ell} &= \Big\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} x^{[1]}(t) \neq 0\Big\},\\
\mathbb{M}^-_{0,0} &= \Big\{x \in \mathbb{M}^- : \lim_{t\to\infty} x(t) = 0, \ \lim_{t\to\infty} x^{[1]}(t) = 0\Big\}.
\end{aligned}$$

Unbounded solutions x of (1.1) are also called either *strongly increasing* (as $t \to \infty$) or *regular increasing* (as $t \to \infty$), according to $x \in \mathbb{M}^+_{\infty,\infty}$ or $x \in \mathbb{M}^+_{\infty,\ell}$, respectively. Such a terminology originates from the Georgian mathematical school, see [18, 22]. Indeed, when $a(t) \equiv 1$, for any unbounded eventually positive solutions x, we have either

$$\lim_{t\to\infty} \frac{x(t)}{t} = \infty \ \text{ or } \ \lim_{t\to\infty} \frac{x(t)}{t} = \ell_x, \ \ 0 < \ell_x < \infty,$$

according to x is strongly increasing or regular increasing, respectively. Analogously, solutions x of (1.1) such that $\lim_{t\to\infty} x(t) = 0$ are called either *strongly decaying* or *regular decaying* (as $t \to \infty$), according to $x \in \mathbb{M}^-_{0,0}$, or $x \in \mathbb{M}^-_{0,\ell}$, respectively. Sometimes, solutions in the subclasses $\mathbb{M}^+_{\infty,\infty}$ and $\mathbb{M}^-_{0,0}$ are called *extremal solutions*.

The following result gives a complete overview of the asymptotic behavior of solutions of (1.1).

Theorem 3.1. *For the half-linear equation* (1.1) *we have* $\mathbb{M}^+ \neq \varnothing$, $\mathbb{M}^- \neq \varnothing$. *Further,*

(1) *If* (C_1) *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \neq \varnothing$ *and* $\mathbb{M}^- = \mathbb{M}^-_{0,0} \neq \varnothing$.

(2) *If* (C_2) *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\ell} \neq \varnothing$ *and* $\mathbb{M}^- = \mathbb{M}^-_{\ell,0} \neq \varnothing$.

(3) *If* (C_3) *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\ell,\infty} \neq \varnothing$ *and* $\mathbb{M}^- = \mathbb{M}^-_{0,\ell} \neq \varnothing$.

(4) *If* (C_4) *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\ell,\ell} \neq \varnothing$, $\mathbb{M}^-_{\ell,\ell} \neq \varnothing$, $\mathbb{M}^-_{\ell,0} \neq \varnothing$, $\mathbb{M}^-_{0,\ell} \neq \varnothing$ *and* $\mathbb{M}^-_{0,0} = \varnothing$.

(5) *If* (C_5) *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\ell} \neq \varnothing$ *and* $\mathbb{M}^- = \mathbb{M}^-_{0,0} \neq \varnothing$.

(6) *If* (C_6) *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \neq \varnothing$ *and* $\mathbb{M}^- = \mathbb{M}^-_{0,\ell} \neq \varnothing$.

(7) *If* (C_7) *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \neq \varnothing$ *and* $\mathbb{M}^- = \mathbb{M}^-_{\ell,0} \neq \varnothing$.

(8) *If* (C_8) *holds, then* $\mathbb{M}^+ = \mathbb{M}^+_{\ell,\infty} \neq \varnothing$ *and* $\mathbb{M}^- = \mathbb{M}^-_{0,0} \neq \varnothing$.

A complete proof of Theorem 3.1 can be found in a forthcoming monograph [8, Chapter V]. It is based on several tools. In particular, we use the Tychonoff fixed point theorem, certain functional integral inequalities jointly with a comparison between (1.1) and the equation

$$\Big(\frac{1}{b^{1/\alpha}(t)}\,|z'|^{1/\alpha}\operatorname{sgn} z'\Big)' - \frac{1}{a^{1/\alpha}(t)}\,|z|^{1/\alpha}\operatorname{sgn} z = 0, \tag{3.1}$$

which comes from (1.1) replacing a by $b^{-1/\alpha}$, b by $a^{-1/\alpha}$ and α with α^{-1}. Equation (3.1) is called *reciprocal equation* to (1.1) and its role in studying the qualitative behavior of solutions of (1.1) is described by the *Reciprocity Principle*, see, e.g., [2, 12]. In particular, observe that the integrals J_1 and J_2 read for (3.1) as Y_2 and Y_1, respectively. Further, also some interesting properties of solutions of (1.1) are also used in the proof, like the property that two solutions of (1.1) can cross at most at one point $T \geq t_0$, whereby the case $T = \infty$ is included, when these solutions are bounded.

Theorem 3.1 extends [3, Theorem 1] by giving the complete classification of solutions. Alternative proofs of some claims of Theorem 3.1 can be found in [3, Theorem 1], too.

4 Examples

Example 4.1. Consider the half-linear equation (1.1) and let there exist $\mu, \nu \in \mathbb{R}$ such that

$$\lim_{t\to\infty}\frac{a(t)}{t^\mu} = a_\infty, \quad \lim_{t\to\infty}\frac{b(t)}{t^\nu} = b_\infty, \quad a_\infty, b_\infty \in (0,\infty). \tag{4.1}$$

Using a standard calculation and Lemma 2.1 we have

$$J_1 = \infty \iff Y_1 = \infty \ \text{ and } \ J_2 = \infty \iff Y_2 = \infty.$$

Consequently, when (4.1) holds, the possible cases concerning the convergence of integrals J_i, Y_i, $i = 1, 2$, are the four cases (C_1)–(C_4). In other words, in this case the integrals Y_1, Y_2 do not play any role. Hence, the situation is exactly the one which happens in the linear case.

Example 4.2. Consider the half-linear equation for $t \ge t_0 > 0$,

$$\big(e^{-3t}(x')^3\big)' - t^{-2}e^{-3t}x^3 = 0. \tag{4.2}$$

For equation (4.2) we have

$$Y_1 = \lim_{T\to\infty} \int_{t_0}^{T} t^{-2}e^{-3t}\bigg(\int_{t}^{T} e^r\,dr\bigg)^3 dt = \infty.$$

Hence, by virtue of Lemma 2.1 we get $J_1 = \infty$. Moreover, we have

$$J_2 = \int_{t_0}^{\infty} e^t\bigg(\int_{t}^{\infty} r^{-2}e^{-3r}\,dr\bigg)^{1/3} dt \ge \int_{t_0}^{\infty}\bigg(\int_{t}^{\infty} r^{-2}\,dr\bigg)^{1/3} dt = \infty, \tag{4.3}$$

and

$$Y_2 = \int_{t_0}^{\infty} t^{-2}e^{-3t}\bigg(\int_{t_0}^{t} e^r\,dr\bigg)^3 dt \le \int_{t_0}^{\infty} t^{-2}\,dt < \infty. \tag{4.4}$$

Thus, for equation (4.2) the case (C_5) holds and so, in view of Theorem 3.1, we obtain $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\ell} \ne \varnothing$ and $\mathbb{M}^- = \mathbb{M}^-_{0,0} \ne \varnothing$.

Example 4.3. Consider the half-linear equation for $t \ge t_0 > 0$,

$$\big(t^{2/3}e^t|x'|^{1/3}\operatorname{sgn} x'\big)' - e^t|x|^{1/3}\operatorname{sgn} x = 0. \tag{4.5}$$

For equation (4.5) we have $J_2 = \infty$. Hence, by virtue of Lemma 2.1 we get $Y_2 = \infty$. Using (4.3) and (4.4) we get $J_1 < \infty$ and $Y_1 = \infty$. Then, for equation (4.5) the case (C_8) holds and so, in view of Theorem 3.1 we obtain $\mathbb{M}^+ = \mathbb{M}^+_{\ell,\infty} \ne \varnothing$ and $\mathbb{M}^- = \mathbb{M}^-_{0,0} \ne \varnothing$. Observe that equation (4.5) is the reciprocal equation to (4.2). Hence, the classification of its solutions can be obtained also using the results in Example 4.2 and the Reciprocity principle.

Example 4.4. Consider the half-linear equations for $t \ge t_0 > 1$,

$$\big(|x'|^{1/2}\operatorname{sgn} x'\big)' - t^{-3/2}(\log t)^{-2/3}|x|^{1/2}\operatorname{sgn} x = 0 \tag{4.6}$$

and

$$\big(t^3(\log t)^{4/3}(x')^2\operatorname{sgn} x'\big)' - x^2\operatorname{sgn} x = 0. \tag{4.7}$$

A standard calculation gives for equation (4.6) that $J_2 < \infty$, $Y_2 = \infty$ and $Y_1 = \infty$. Thus, Lemma 2.1 yields $J_1 = \infty$ and the case (C_7) holds. Applying Theorem 3.1 we obtain for equation (4.6) $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \ne \varnothing$ and $\mathbb{M}^- = \mathbb{M}^-_{\ell,0} \ne \varnothing$.

Now, consider equation (4.7). Since this equation is the reciprocal equation to (4.6), for equation (4.7) we have $J_1 = J_2 = Y_2 = \infty$ and $Y_1 < \infty$. Thus, for equation (4.7) the case (C_6) holds and by Theorem 3.1 we get $\mathbb{M}^+ = \mathbb{M}^+_{\infty,\infty} \ne \varnothing$ and $\mathbb{M}^- = \mathbb{M}^-_{0,\ell} \ne \varnothing$.

Some applications of Theorem 3.1 to the nonlinear differential equation

$$\big(a(t)|x'|^{\alpha}\operatorname{sgn} x'\big)' - \widetilde{b}(t)F(x) = 0,$$

where the weight $\widetilde{b}$ has indefinite sign and F is a continuous function on $\mathbb{R}$ such that $uF(u) > 0$ for $u \ne 0$, can be found in [9, 10].

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On Asymptotics of Rapidly Varying Solutions of Non-Autonomous Differential Equations of Third-Order

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Consider the differential equation

$$y''' = \alpha_0 p(t)\varphi(y), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $y < a < \omega \le +\infty$, $\varphi : \Delta_{Y_0} \to]0, +\infty[$ is a continuously differentiable function such that

$$\varphi'(y) \neq 0 \ \text{ for } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to 0 \\ y \in \Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{or} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi(y)\varphi''(y)}{\varphi'^2(y)} = 1, \tag{2}$$

Y_0 equals either zero or $\pm\infty$, Δ_{Y_0} is some one-sided neighborhood of Y_0.

From the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} + 1 \ \text{ for } \ y \in \Delta_{Y_0}$$

and conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ for } \ y \to Y_0 \quad \big(y \in \Delta_{Y_0}\big) \ \text{ and } \ \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

This means that in the considered equation the continuous function φ and its first order derivatives are (see [8, Ch. 3, § 3.4, Lemmas 3.2, 3.3, pp. 91–92]) rapidly changing as $y \to Y_0$.

For two-term differential equations of the form (1) with nonlinearities satisfying condition (2), the asymptotic properties of solutions were studied in the works of M. Maric [8], V. M. Evtukhov and his students N. G. Drik, V. M. Kharkov, A. G. Chernikova [3–5].

In the works of V. M. Evtukhov, A. G. Chernikova [3] for the differential equation (1) of the second order in the case, when φ is a rapidly changing function as $t \to +\infty$, the asymptotic properties of the so-called $P_\omega(Y_0, \lambda_0)$-solutions were studied. In this work, we propose the distribution of these results to third-order differential equations.

Definition 1. Solution y of equation (1) is called $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if it is specified on the interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions

$$y(t) \in \Delta_{Y_0}, \ \text{ where } \ t \in [t_0, \omega[\,,$$

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad k = 1, 2, \quad \lim_{t \uparrow \omega} \frac{y''^2(t)}{y'''(t)y'(t)} = \lambda_0.$$

The goal of this work is to establish the necessary and sufficient conditions for the existence for equation (1) of (Y_0,λ_0)-solutions in the non-singular case, when $\lambda_0\in\mathbb{R}\setminus\{0,1,\frac{1}{2}\}$, and in the singular case, when $\lambda_0=1$, as well as asymptotic for $t\uparrow\omega$ representations for such solutions and their derivatives up to the second order.

Without loss of generality, we will further assume that

$$\Delta_{Y_0}=\begin{cases}[y_0,Y_0[& \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0,y_0] & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0,\end{cases}$$

where $y_0\in\mathbb{R}$ is such that $|y_0|<1$ when $Y_0=0$, and $y_0>1$ $(y_0<-1)$, when $Y_0=+\infty$ (when $Y_0=-\infty$).

The function $f:\Delta_{Y_0}\to\mathbb{R}\setminus\{0\}$, satisfying condition (2), when $Y_0=\pm\infty$, and $\lim\limits_{y\to+\infty}f(y)=+\infty$, belongs to the class $\Gamma_{Y_0}(Z_0)$ of the functions $\varphi:\Delta_{Y_0}\to]0,+\infty[$, where Y_0 equals either zero or $\pm\infty$, and Δ_{Y_0} is a one-sided neighborhood of Y_0, for which

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\varphi(y)=Z_0=\begin{cases} & 0,\\ \text{or} & +\infty,\end{cases}\tag{3}$$

which extends the class of function Γ, introduced by L. Khan (see, for example, [6, Ch. 3, p. 3.10, p. 175]).

If $f\in\Gamma_{Y_0}(Z_0)$ with the complementary function g, and, moreover, is continuous and strictly monotone, then there exists a continuous strictly monotone inverse function $f^{-1}:\Delta_{Z_0}\longrightarrow\Delta_{Y_0}$, where

$$\Delta_{Z_0}=\begin{cases} & [z_0,Z_0[\,,\\ \text{or} &]Z_0,z_0],\end{cases}\quad z_0=f(y_0),\quad Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}f(y).$$

We introduce the necessary auxiliary notation. We assume that the domain of the function φ in equation (1) is determined by formula (3). Next, we set

$$\mu_0=\operatorname{sign}\varphi'(y),\quad \nu_0=\operatorname{sign}y_0,\quad \nu_1=\begin{cases}1 & \text{if } \Delta_{Y_0}=[y_0,Y_0[\,,\\ -1 & \text{if } \Delta_{Y_0}=]Y_0,y_0],\end{cases}$$

and introduce the functions

$$J(t)=\int\limits_A^t\pi_\omega^2(\tau)p(\tau)\,d\tau,\quad \Phi(y)=\int\limits_B^y\frac{ds}{\varphi(s)}\,,$$

where

$$\pi_\omega=\begin{cases}t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty,\end{cases}$$

$$A=\begin{cases}\omega & \text{if } \int\limits_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=const,\\ a & \text{if } \int\limits_a^\omega\pi_\omega^2(\tau)p(\tau)\,d\tau=\infty,\end{cases}\qquad B=\begin{cases}Y_0 & \text{if } \int\limits_{y_0}^{Y_0}\dfrac{ds}{\varphi(s)}=const,\\ y_0 & \text{if } \int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=const.\end{cases}$$

Considering the definition of $P_\omega(Y_0,\lambda_0)$-solutions of the differential equation (1), we note that the numbers ν_0, ν_1, ν_2 and α_0 determine the signs of any $P_\omega(Y_0,\lambda_0)$-solutions of its first, second and third derivatives (respectively) in some left neighborhood of ω. It is clear that the condition

$$\nu_0\nu_1<0, \ \text{ if } \ Y_0=0, \quad \nu_0\nu_1>0, \ \text{ if } \ Y_0=\pm\infty,$$

is necessary for the existence of such solutions.

Now we turn our attention to some properties of the function Φ. It retains a sign on the interval Δ_{Y_0}, tends either to zero or $\pm\infty$ when $y\to Y_0$ and increasing by Δ_{Y_0}, because on this interval $\Phi'(y)=\frac{1}{\varphi(y)}>0$. Therefore, for it there is an inverse function $\Phi^{-1}:\Delta_{Z_0}\to\Delta_{Y_0}$, where due to the second of conditions (2) and the monotone increase of Φ^{-1},

$$Z_0=\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\Phi(y)=\begin{cases} \quad 0, \\ \text{or} \quad +\infty, \end{cases} \qquad \Delta_{Z_0}=\begin{cases} [z_0,Z_0[& \text{for } \ \Delta_{Y_0}=[y_0,Y_0[\,, \\ ]Z_0,z_0] & \text{for } \ \Delta_{Y_0}=]Y_0,y_0], \end{cases} \qquad z_0=\varphi(y_0).$$

For $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac{1}{2}\}$ we also introduce auxiliary functions:

$$q(t)=\frac{\alpha_0(\lambda_0-1)^2\pi_\omega^3(t)p(t)\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,(\lambda_0-1)J(t))\big)}{\lambda_0\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}J(t)\big)}\,,$$

$$H(t)=\frac{\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\varphi'\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}{\varphi\big(\Phi^{-1}(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t))\big)}\,,$$

For equation (1) the following assertions take place.

Theorem 1. *Let $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac{1}{2}\}$. Then for the existence for the differential equation* (1) *of $P_\omega(Y_0,\lambda_0)$-solutions, it is necessary to comply with the conditions*

$$\alpha_0\nu_1\lambda_0>0, \quad \nu_0\nu_1(2\lambda_0-1)(\lambda_0)\pi_\omega(t)>0 \ \textit{ and } \ \alpha_0\mu_0\lambda_0J(t)<0 \ \textit{ for } \ t\in(a,\omega), \tag{4}$$

$$\frac{\alpha_0}{\lambda_0}\lim_{t\uparrow\omega}J(t)=Z_0, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'(t)}{J(t)}=\pm\infty, \quad \lim_{t\uparrow\omega}q(t)=\frac{2\lambda_0-1}{\lambda_0-1}\,. \tag{5}$$

Moreover, for each such solution, the following asymptotic representations take place:

$$y(t)=\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big] \ \textit{ for } \ t\uparrow\omega, \tag{6}$$

$$y'(t)=\frac{(2\lambda_0-1)}{(\lambda_0-1)}\,\frac{\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}{\pi_\omega(t)}\,[1+o(1)] \ \textit{ for } \ t\uparrow\omega, \tag{7}$$

$$y''(t)=\frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)^2}\,\frac{\Phi^{-1}\big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\big)}{\pi_\omega^2(t)}\,[1+o(1)] \ \textit{ for } \ t\uparrow\omega.$$

Theorem 2. *Let $\lambda_0\in\mathbb{R}\setminus\{0;1;\frac{1}{2}\}$, conditions* (4), (5) *met, there exist a finite or equal to $\pm\infty$ limit*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2}\,\sqrt[3]{\Big(\frac{y\varphi'(y)}{\varphi(y)}\Big)^2},$$

and there exist the limit

$$\lim_{t\uparrow\omega}\Big[\frac{2\lambda_0-1}{\lambda_0-1}-q(t)\Big]|H(t)|^{\frac{2}{3}}=0,$$

Then, the differential equation (1) *has at least one* $P_\omega(Y_0,\lambda_0)$*-solution, which allows for* $t\uparrow\omega$ *the asymptotic representations*

$$
\begin{gathered}
y(t)=\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)\Big[1+\frac{o(1)}{H(t)}\Big],\\
y'(t)=\frac{2\lambda_0-1}{(\lambda_0-1)\pi_\omega(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+o(1)H^{-\frac{2}{3}}],\\
y''(t)=\frac{\lambda_0(2\lambda_0-1)}{(\lambda_0-1)\pi_\omega^2(t)}\,\Phi^{-1}\Big(\alpha_0\,\frac{(\lambda_0-1)^2}{\lambda_0}\,J(t)\Big)[1+o(1)H^{-\frac{1}{3}}],
\end{gathered}
\tag{8}
$$

and in the case when

$$\mu_0\lambda_0(2\lambda_0-1)(\lambda_0-1)<0\ \ \textit{for}\ \ t\in(a,\omega),$$

the differential equation (1) *has a one-parameter family of* $P_\omega(Y_0,\lambda_0)$*-solutions, but in the case when*

$$\mu_0\lambda_0(2\lambda_0-1)(\lambda_0-1)>0\ \ \textit{for}\ \ t\in(a,\omega),$$

the differential equation (1) *has a two-parameter family of* $P_\omega(Y_0,\lambda_0)$*-solutions with representations* (6), (7), *and such that the first and second order derivatives allow the asymptotic representations* (8).

Introduce the functions

$$J_1(t)=\int\limits_{A_1}^{t}p^{\frac{1}{3}}(\tau)\,d\tau,\quad \Phi_1(y)=\int\limits_{B_1}^{y}\frac{ds}{|s|^{\frac{2}{3}}\varphi^{\frac{1}{3}}(s)},$$

where

$$A_1=\begin{cases}\omega & \text{if}\ \displaystyle\int\limits_a^\omega p^{\frac{1}{3}}(\tau)\,d\tau<+\infty,\\ a & \text{if}\ \displaystyle\int\limits_a^\omega p^{\frac{1}{3}}(\tau)\,d\tau=+\infty,\end{cases}\qquad B_1=\begin{cases}Y_0 & \text{if}\ \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac{2}{3}}\varphi^{\frac{1}{3}}(s)}=const,\\ y_0 & \text{if}\ \displaystyle\int\limits_{y_0}^{Y_0}\frac{ds}{|s|^{\frac{2}{3}}\varphi^{\frac{1}{3}}(s)}=\pm\infty.\end{cases}$$

Consider the definition of $P_\omega(Y_0,1)$-solutions of the differential equation (1). It is clear that the conditions

$$\nu_0\nu_1<0,\ \text{if}\ Y_0=0,\quad \nu_0\nu_1>0,\ \text{if}\ Y_0=\pm\infty,$$

and

$$\nu_1\alpha_0<0,\ \text{for}\ \lim_{t\uparrow\omega}y'(t)=0,\quad \nu_1\alpha_0>0,\ \text{for}\ \lim_{t\uparrow\omega}y'(t)=\pm\infty,$$

are necessary for the existence of such solutions. For $\lambda_0=1$, we also introduce the auxiliary functions

$$
\begin{gathered}
q_1(t)=\frac{\alpha_0\nu_1J_3(t)}{p^{\frac{1}{3}}(t)\Phi_1^{-1}(\nu_1J_1(t))^{\frac{2}{3}}\varphi^{\frac{1}{3}}(\Phi_1^{-1}(\nu_1J_1(t)))}\,,\\
H_1(t)=\frac{\Phi_1^{-1}(\nu_1J_1(t))\varphi'(\Phi_1^{-1}(\nu_1J_1(t)))}{\varphi(\Phi_1^{-1}(\nu_1J_1(t)))}\,,\\
J_2(t)=\int\limits_{A_2}^{t}p(\tau)\varphi(\Phi_1^{-1}(\nu_1J_1(\tau)))\,d\tau,\quad J_3(t)=\int\limits_{A_3}^{t}J_2(\tau)\,d\tau,
\end{gathered}
$$

where

$$A_2=\begin{cases} t_0 & \text{if } \int\limits_{t_2}^{\omega} p(\tau)\varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_{t_2}^{\omega} p(\tau)\varphi(\Phi_1^{-1}(\nu_1 J_1(\tau)))\,d\tau<+\infty,\end{cases}\qquad A_3=\begin{cases} t_0 & \text{if } \int\limits_{t_3}^{\omega} J_2(\tau)\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_{t_3}^{\omega} J_2(\tau)\,d\tau<+\infty,\end{cases}$$

$$t_2,t_3\in[a,\omega].$$

For equation (1) the following assertions take place.

Theorem 3. *For the existence for the differential equation* (1) *of* $P_\omega(Y_0,1)$*-solutions it is necessary to comply with the conditions*

$$\alpha_0\nu_0>0,\ \ \mu_0\nu_1 J_1(t)<0\ \ \textit{for}\ \ t\in\,]a,\omega[\,,$$

$$\nu_1\lim_{t\uparrow\omega} J_1(t)=Z_0,\quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)J_1'(t)}{J_1(t)}=\pm\infty,\quad \lim_{t\uparrow\omega} q_1(t)=1,\ \ \lim_{t\uparrow\omega}\frac{p(t)\varphi(\Phi_1^{-1}(\nu_1 J_1(t)))J_3(t)}{(J_2(t))^2}=1.$$

Moreover, for each solution, there take place the asymptotic representations for $t\uparrow\omega$

$$\begin{aligned} y(t)&=\Phi_1^{-1}\big(\alpha_0(\lambda_0-1)J_1(t)\big)\Big[1+\frac{o(1)}{H_1(t)}\Big],\\ y'(t)&=\nu_1 p^{\frac13}(t)\varphi^{\frac13}\big(\Phi_1^{-1}(\nu_1 J_1(t))\big)\big(\Phi_1^{-1}(\nu_1 J_1(t))\big)^{\frac23}[1+o(1)],\\ y''(t)&=\alpha_0 J_2(t)[1+o(1)].\end{aligned}$$

Similarly to Theorem 2, we prove a sufficient condition for the existence of $P_\omega(Y_0,1)$-solutions.

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On an Upper Estimate for the First Eigenvalue of a Sturm–Liouville Problem

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1 Introduction

Consider the Sturm–Liouville problem

$$y'' + Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1.1}$$

$$y(0) = y(1) = 0, \tag{1.2}$$

where Q belongs to the set $T_{\alpha,\beta,\gamma}$ of all measurable locally integrable on $(0,1)$ functions with non-negative values such that the following integral conditions hold

$$\int_0^1 x^{\alpha}(1-x)^{\beta}Q^{\gamma}(x)\,dx = 1, \quad \alpha,\beta,\gamma \in \mathbb{R}, \quad \gamma \neq 0, \tag{1.3}$$

$$\int_0^1 x(1-x)Q(x)\,dx < \infty.$$

A function y is a *solution to problem* (1.1), (1.2) if it is absolutely continuous on the segment $[0,1]$, satisfies (1.2), its derivative y' is absolutely continuous on any segment $[\rho, 1-\rho]$, where $0 < \rho < \frac{1}{2}$, and equality (1.1) holds almost everywhere in the interval $(0,1)$.

This work is a continuation of studies of estimates for the first eigenvalue of the Sturm–Liouville problem with the equation $y'' + \lambda Q(x)y = 0$, Dirichlet boundary conditions, and a non-negative summable on $[0,1]$ potential Q satisfying the condition $\|Q\|_{L_{\gamma}(0,1)} = 1$, $\gamma \neq 0$, initiated by Y. V. Egorov and V. A. Kondratiev in [1]. We study a problem of that kind provided the integral conditions contain weight functions.

For $\gamma < 0$, $\alpha \leq 2\gamma - 1$, $-\infty < \beta < +\infty$ or $\gamma < 0$, $\beta \leq 2\gamma - 1$, $-\infty < \alpha < +\infty$, the set $T_{\alpha,\beta,\gamma}$ is empty and the first eigenvalue of problem (1.1), (1.2) does not exist. For other values of $\alpha, \beta, \gamma, \gamma \neq 0$, denote

$$M_{\alpha,\beta,\gamma} = \sup_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q).$$

2 Main results

It [2], the following theorem was proved.

Theorem 2.1. *If $0 < \gamma < 1$, $\alpha, \beta > 2\gamma - 1$, then $M_{\alpha,\beta,\gamma} < \pi^2$.*

In the proof of this theorem it was supposed that for any $0<\gamma<1$, $\alpha,\beta>2\gamma-1$ we have $M_{\alpha,\beta,\gamma}=\pi^2$, that is, for any sufficiently small $\varepsilon>0$ there exists a function $Q\in T_{\alpha,\beta,\gamma}$ such that $\lambda_1(Q)>(\pi-\varepsilon)^2$. Under this assumption we got a contradiction with condition (1.3), namely, it was proved that in this case there exists a positive constant C, depending on α, β, γ, such that

$$\int\limits_0^1 x^\alpha(1-x)^\beta Q^\gamma(x)dx\le C\varepsilon^M,$$

where

$$M=\min\Big\{\frac{(\alpha-2\gamma+1)\gamma}{1-\gamma+\alpha}\,,\frac{(\beta-2\gamma+1)\gamma}{1-\gamma+\beta}\Big\}>0.$$

Let us prove the following

Theorem 2.2. *If $\alpha,\beta>1$, then $M_{\alpha,\beta,1}<\pi^2$.*

Proof. Suppose that $M_{\alpha,\beta,1}=\pi^2$, $\alpha,\beta>1$.

Let $0<\gamma<1$. By the Hölder inequality, we have

$$\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^\gamma(x)\,dx\leqslant\bigg(\int\limits_0^1 x^\alpha(1-x)^\beta Q(x)\,dx\bigg)^\gamma.$$

Then

$$\bigg(\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^\gamma(x)\,dx\bigg)^{\frac{1}{\gamma}}\leqslant\int\limits_0^1 x^\alpha(1-x)^\beta Q(x)\,dx=1. \tag{2.1}$$

Note that, for $0<\gamma<1$, the inequality $\alpha\gamma>2\gamma-1$ holds if and only if $\alpha>1$. Similarly, $\beta\gamma>2\gamma-1$ if and only if $\beta>1$.

Denote by $\widetilde{T}_{\alpha\gamma,\beta\gamma,\gamma}$ a set of measurable non-negative locally integrable on $(0,1)$ functions Q such that

$$\bigg(\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^\gamma(x)\,dx\bigg)^{\frac{1}{\gamma}}\leqslant 1.$$

By virtue of (2.1),

$$T_{\alpha,\beta,1}\subset\widetilde{T}_{\alpha\gamma,\beta\gamma,\gamma}.$$

If we suppose that

$$M_{\alpha,\beta,1}=\sup_{Q\in T_{\alpha,\beta,1}}\lambda_1(Q)=\pi^2,$$

then for $0<\gamma<1$, we also have

$$M_{\alpha\gamma,\beta\gamma,\gamma}=\sup_{Q\in\widetilde{T}_{\alpha\gamma,\beta\gamma,\gamma}}\lambda_1(Q)=\pi^2.$$

If $M_{\alpha,\beta,1}=\sup\limits_{Q\in T_{\alpha,\beta,1}}\lambda_1(Q)=\pi^2$, then for any $\varepsilon>0$ there exists a function $Q_*\in T_{\alpha,\beta,1}$ such that

$$\lambda_1(Q_*)>(\pi-\varepsilon)^2.$$

This function Q_* belongs to $\widetilde{T}_{\alpha\gamma,\beta\gamma,\gamma}$ either and following the proof of Theorem 2.1, we can find a positive constant C, depending on α,β,γ, such that

$$\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}{Q_*}^{\gamma}(x)\,dx \leqslant C\varepsilon^M,$$

where

$$M=\min\Big\{\frac{(\alpha\gamma-2\gamma+1)\gamma}{1-\gamma+\alpha\gamma}\,,\frac{(\beta\gamma-2\gamma+1)\gamma}{1-\gamma+\beta\gamma}\Big\}.$$

Suppose that $M=\frac{(\alpha\gamma-2\gamma+1)\gamma}{1-\gamma+\alpha\gamma}$ (in case $M=\frac{(\beta\gamma-2\gamma+1)\gamma}{1-\gamma+\beta\gamma}$ the proof is similar). For any $0<\gamma<1$, $\alpha>1$, we have $\alpha\gamma>2\gamma-1$ and M is positive.

Note that for a fixed $\alpha>1$, if γ approaches 1, the exponent of $\varepsilon^{\frac{(\alpha\gamma-2\gamma+1)\gamma}{1-\gamma+\alpha\gamma}}$ approaches a concrete positive number $\frac{\alpha-1}{\alpha}$.

As soon as γ tends to 1, the factor C, depending on α,β,γ, tends to some constant $\widetilde{C}$. Let us choose ε in such a way that the following inequality

$$\widetilde{C}\varepsilon^{\frac{\alpha-1}{\alpha}}<\frac{1}{2}$$

holds.

Then we get a contradiction

$$1=\int\limits_0^1 x^{\alpha}(1-x)^{\beta}Q(x)\,dx=\lim_{\gamma\to1}\bigg(\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^{\gamma}(x)\,dx\bigg)^{\frac{1}{\gamma}}\leqslant\widetilde{C}\varepsilon^{\frac{\alpha-1}{\alpha}}<\frac{1}{2}\,.$$

Note that, while γ increases from 0 to 1, the integral $\big(\int\limits_0^1 x^{\alpha\gamma}(1-x)^{\beta\gamma}Q^{\gamma}(x)\,dx\big)^{\frac{1}{\gamma}}$ also increases. Indeed, if $\gamma_1<\gamma_2$, then by virtue of the Hölder inequality, since $\frac{\gamma_2}{\gamma_1}>1$, we have

$$\int\limits_0^1 x^{\alpha\gamma_1}(1-x)^{\beta\gamma_1}Q^{\gamma_1}(x)\,dx\leqslant\bigg(\int\limits_0^1 x^{\alpha\gamma_2}(1-x)^{\beta\gamma_2}Q^{\gamma_2}(x)\,dx\bigg)^{\frac{\gamma_1}{\gamma_2}}$$

and

$$\bigg(\int\limits_0^1 x^{\alpha\gamma_1}(1-x)^{\beta\gamma_1}Q^{\gamma_1}(x)\,dx\bigg)^{\frac{1}{\gamma_1}}\leqslant\bigg(\int\limits_0^1 x^{\alpha\gamma_2}(1-x)^{\beta\gamma_2}Q^{\gamma_2}(x)\,dx\bigg)^{\frac{1}{\gamma_2}}.\qquad\square$$

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On Some Bounds for Coefficients of the Asymptotics to Robin Eigenvalue

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Let us consider the Robin eigenvalue problem

$$\Delta u+\lambda u=0,\ \ x\in\Omega, \tag{1}$$

$$\Big(\frac{\partial u}{\partial\nu}+\alpha u\Big)\Big|_{x\in\Gamma}=0,\ \ \alpha\in\mathbb{R}. \tag{2}$$

Here $\Omega\subset\mathbb{R}^n$, $n\geq 2$, is a bounded domain with the sufficiently smooth boundary Γ. We denote by $\lambda_1^R(\alpha)$ the first eigenvalue of problem (1), (2). Consider also the Dirichlet eigenvalue problem

$$\Delta u+\lambda u=0,\ \ x\in\Omega, \tag{3}$$

$$u\big|_{x\in\Gamma}=0. \tag{4}$$

Let λ_1^D be the first eigenvalue of problem (3), (4), and $u_1^D(x)$ be the first Dirichlet eigenfunction, satisfying $\|u_1^D\|_{L_2(\Omega)}=1$.

In the papers [1–5] we get the following statement.

Theorem 1. *The eigenvalue $\lambda_1^R(\alpha)$ satisfies the asymptotic representation*

$$\lambda_1^R(\alpha)=\lambda_1^D-a_1\alpha^{-1}-a_2\alpha^{-2}+o(\alpha^{-2}),\ \ \alpha\to+\infty, \tag{5}$$

$$a_1=\int\limits_\Gamma\Big(\frac{\partial u_1^D}{\partial\nu}\Big)^2\,ds,\quad a_2=\int\limits_\Gamma\frac{\partial u_1^D}{\partial\nu}\,\frac{\partial v}{\partial\nu}\,ds. \tag{6}$$

The function $v\in H^1(\Omega)$ is a solution of the boundary value problem

$$\Delta v+\lambda_1^D v=\int\limits_\Gamma\Big(\frac{\partial u_1^D}{\partial\nu}\Big)^2\,ds\,u_1^D,\ \ x\in\Omega, \tag{7}$$

$$v\big|_{x\in\Gamma}=-\frac{\partial u_1^D}{\partial\nu}\Big|_{x\in\Gamma}, \tag{8}$$

satisfying the condition

$$\int\limits_\Omega vu_1^D\,dx=0. \tag{9}$$

Problem (7)–(9) *has a unique solution.*

In this paper we establish two-sided estimates for the coefficient a_1 in formula (5).

Theorem 2. *Let $\Omega \subset B_{R_0}(0) = \{x \in \mathbb{R}^n : |x| < R_0\}$ and $\boldsymbol{b}(x) = (b_1(x), \dots, b_n(x)) \in C^1(\overline{\Omega})$ be a vector function. Then the following estimates hold:*

$$\frac{2\lambda_1^D}{R_0} \le a_1 \le 4n \inf_{\substack{\boldsymbol{b} \in C^1(\overline{\Omega}) \\ \boldsymbol{b}|_\Gamma = \nu}} \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \lambda_1^D, \ \ \|f(x)\|_{C(\overline{\Omega})} = \sup_{x \in \overline{\Omega}} |f(x)|. \tag{10}$$

Definition. We call Γ a strictly star-shaped surface if the inequality $(\nu, x) > 0$ holds for all $x \in \Gamma$.

Theorem 3. *Let Γ be a strictly star-shaped surface. Then the following estimate holds:*

$$a_1 \le \frac{2\lambda_1^D}{\inf\limits_{x \in \Gamma} (\nu, x)} \, . \tag{11}$$

Let us note that for $\Omega = B_{R_0}(0)$ it follows from (10), (11) that $a_1 = \frac{2\lambda_1^D}{R_0}$.

Proof. By direct computation we have the following equality for solutions of problem (3), (4):

$$\int\limits_\Gamma (\mathbf{b}, \nu) u_\nu^2 \, ds = \int\limits_\Omega \Big(2 \sum_{i,j=1}^n (b_i)_{x_j} u_{x_i} u_{x_j} + \operatorname{div} \mathbf{b} \big(\lambda u^2 - |\nabla u|^2 \big) \Big) \, dx. \tag{12}$$

Using (12) for $\mathbf{b}|_\Gamma = \nu$, we get

$$\int\limits_\Gamma u_\nu^2 \, ds = \int\limits_\Gamma (\mathbf{b}, \nu) u_\nu^2 \, ds \le 2 \int\limits_\Omega \sum_{i,j=1}^n (b_i)_{x_j} u_{x_i} u_{x_j} \, dx + \int\limits_\Omega |\operatorname{div} \mathbf{b}| \big(|\nabla u|^2 + \lambda u^2 \big) \, dx. \tag{13}$$

We have

$$\begin{aligned} \sum_{i,j=1}^n (b_i)_{x_j} u_{x_i} u_{x_j} &\le \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \sum_{i,j=1}^n |u_{x_i}| \, |u_{x_j}| \\ &= \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \Big(\sum_{i=1}^n |u_{x_i}| \Big)^2 \le n \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} |\nabla u|^2, \ \ x \in \Omega. \end{aligned} \tag{14}$$

Now, combine (13), (14) and the inequality

$$|\operatorname{div} \mathbf{b}| \le n \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})}, \ \ x \in \Omega,$$

we get

$$\int\limits_\Gamma u_\nu^2 \, ds \le \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \Big(3n \int\limits_\Omega |\nabla u|^2 \, dx + \lambda \int\limits_\Omega u^2 \, dx \Big). \tag{15}$$

It follows from (3), (4) that

$$\int\limits_\Omega |\nabla u|^2 \, dx = \lambda \int\limits_\Omega u^2 \, dx. \tag{16}$$

Therefore, by (15) and (16),

$$\int\limits_\Gamma u_\nu^2 \, ds \le 4n \max_{i,j=1,\dots,n} \|(b_i)_{x_j}\|_{C(\overline{\Omega})} \lambda \int\limits_\Omega u^2 \, dx. \tag{17}$$

Taking $u = u_1^D$ with $\|u_1^D\|_{L_2(\Omega)} = 1$, we get from (6) and (17) the upper estimate (10).

Let us prove now the lower estimate (10). We have the Rellich equality for normalized in $L_2(\Omega)$ eigenfunctions of problem (3), (4) (see [6, 7]):

$$\lambda = \frac{1}{2} \int\limits_{\Gamma} (x, \nu) u_\nu^2 \, ds. \tag{18}$$

Therefore,

$$2\lambda = \int\limits_{\Gamma} (x, \nu) u_\nu^2 \, ds \le \int\limits_{\Gamma} |x| u_\nu^2 \, ds \le \sup_{x \in \Gamma} \int\limits_{\Gamma} u_\nu^2 \, ds \le R_0 \int\limits_{\Gamma} u_\nu^2 \, ds.$$

Now, for $u = u_1^D$ we obtain

$$a_1 \ge \frac{2\lambda_1^D}{R_0}. \qquad \square$$

The proof of Theorem 3 is based on the Rellich equality (18) for u_1^D in a strictly star-shaped domain Ω.

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Approximate Optimal Control on Semi-Axis for the Reaction-Diffusion Process with Fast-Oscillating Coefficients

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1 Introduction and setting of the problem

It is known that the local characteristics of processes in micro-inhomogeneous medium contain functions of the form $a(\frac{x}{\varepsilon})$, where $\varepsilon > 0$ is a small parameter. Passing to averaged parameters is an effective tool for studying such processes [10]. Such a procedure for parabolic operators was justified in [7]. Optimal control problems for parabolic equations with fast oscillating functions in the coefficients were investigated in [1,6,12,14]. General questions of the solvability of systems of the reaction-diffusion type were investigated in [4,9,13]. In this paper, we consider the optimal control problem on semi-axis for the reaction - diffusion equation with a coercive objective functional, whose coefficients contain fast oscillating functions.

More precisely, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\varepsilon \in (0,1)$ be a small parameter. In the cylinder $Q = (0,\infty) \times \Omega$, the controlled process is described by the evolutionary system

$$\begin{cases} \dfrac{\partial y}{\partial t} = \operatorname{div}\Big[a\Big(\dfrac{x}{\varepsilon}\Big)\nabla y\Big] - b\Big(\dfrac{x}{\varepsilon}\Big)f(y) + u(t,x), \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{t=0} = y_0^{\varepsilon}(x), \end{cases} \tag{1.1}$$

$$u \in U \subseteq L^2(Q), \tag{1.2}$$

$$J(y,u) = \int\limits_Q q_\varepsilon(t,x,y(t,x))y(t,x)\,dt\,dx + \gamma \int\limits_Q u^2(t,x)\,dt\,dx \longrightarrow \inf, \ \ \gamma > 0. \tag{1.3}$$

Under natural assumptions on parameters we prove the following limit equality

$$J(\overline{y}^{\,\varepsilon}, \overline{u}^{\,\varepsilon}) \longrightarrow J(\overline{y}, \overline{u}), \ \ \varepsilon \to 0,$$

where $\{\overline{y}^{\,\varepsilon}, \overline{u}^{\,\varepsilon}\}$ and $\{\overline{y}, \overline{u}\}$ are optimal processes of the perturbed problem (1.1)–(1.3) and the corresponding averaged problem.

2 Main results

We will consider the optimal control problem (1.1)–(1.3) under following assumptions:

a is a measurable, periodic, symmetric matrix satisfying the condition of uniform ellipticity and boundedness

$$\forall\, x \in \mathbb{R}^n, \ \ \forall\, \eta \in \mathbb{R}^n \ \ \nu_1 \sum_{i=1}^n \eta_i^2 \le \sum_{i,j=1}^n a_{ij}(x)\eta_i\eta_j \le \nu_2 \sum_{i=1}^n \eta_i^2, \tag{2.1}$$

$b \in L^\infty(\mathbb{R}^n)$ is non-negative, bounded, periodic function,

$$\exists\, b_1 > 0, \ \ \exists\, b_0 > 0, \ \ \forall\, s \in \mathbb{R} \ \ b_0 \le b(s) \le b_1, \tag{2.2}$$

nonlinearity $f \in C(\mathbb{R})$ satisfies the standard conditions of sign and growth:

$$\exists\, \alpha > 0, \ \ C \ge 0, \ \ p \ge 2: \ \ \forall\, s \in \mathbb{R}, \ \ f(s) \cdot s \ge \alpha |s|^p, \ \ |f(s)| \le C\big(1 + |s|^{p-1}\big), \tag{2.3}$$

$$U \text{ is convex, closed set in } L^2(Q),\ 0 \in U.$$

The function $q_\varepsilon : Q \times \mathbb{R} \longmapsto \mathbb{R}$ is a Carathéodory function, and there exist a constant $K > 0$ independent of $\varepsilon \in (0,1)$ and non-negative functions $K_1 \in L^1(Q)$, $K_2 \in L^2(Q)$ such that

$$q_\varepsilon(t,x,\xi)\xi \ge -K_1(t,x), \ \ |q_\varepsilon(t,x,\xi)| \le K|\xi| + K_2(t,x). \tag{2.4}$$

Under conditions (2.1)–(2.3), it is known [13] that for any $\varepsilon > 0$, $\forall\, u \in L^2(Q)$, $\forall\, y_0^\varepsilon \in L^2(\Omega)$, problem (1.1) has at least one solution $y = y(t,x)$ in the class

$$W := \Big\{ y \in L^2(0,\infty; H_0^1(\Omega)) : \ \frac{dy}{dt} \in L^2(0,\infty; H^{-1}(\Omega)) \Big\}.$$

Moreover, each solution (1.1) from W belongs to $C([0,\infty); L^2(\Omega))$.

Theorem 2.1. *Under conditions* (2.1)–(2.4) *the optimal control problem* (1.1)–(1.3) *has a solution* $\{\overline{y}^{\,\varepsilon}, \overline{u}^{\,\varepsilon}\}$.

Now let us discuss averaged problem ($\varepsilon = 0$).

We assume that a constant, positive defined matrix $\widehat{a}$ is averaged for $a(\frac{x}{\varepsilon})$ [10], the number $\widehat{b}$ is the mean value of a periodic function $b(x)$, and there exists a Carathéodory function $q_\varepsilon : Q \times \mathbb{R} \longmapsto \mathbb{R}$ such that

$$\forall\, r > 0 \ \ q_\varepsilon(t,x,\xi) \to q(t,x,\xi) \ \text{ weakly in } L^2(Q) \text{ uniformly with respect to } \ |\xi| \le r. \tag{2.5}$$

Consider problem (1.1)–(1.3) with averaged coefficients

$$\begin{cases} \dfrac{\partial y}{\partial t} = \operatorname{div}[\widehat{a} \nabla y] - \widehat{b} f(y) + u(t,x), \\ y\big|_{\partial\Omega} = 0, \\ y\big|_{t=0} = y_0(x), \end{cases} \tag{2.6}$$

$$u \in U \subseteq L^2(Q_T), \tag{2.7}$$

$$J(y,u) = \int\limits_Q q(t,x,(t,x)) y(t,x)\, dx + \gamma \int\limits_Q u^2(t,x)\, dt\, dx \longrightarrow \inf. \tag{2.8}$$

Using convergence (2.5), it is easy to show that the function $q_\varepsilon : Q \times \mathbb{R} \mapsto \mathbb{R}$ satisfies inequalities (2.4). Then, by Theorem 2.1, we can assert that problem (2.6)–(2.8) has a solution $\{\overline{y}, \overline{u}\}$.

We will assume the following additional condition:

$$\text{for any } \ u \in U \ \text{ problem (2.6) has a unique solution.} \tag{2.9}$$

Condition (2.9) will take place if $f \in C^1(\mathbb{R})$ and $f'(s) \ge -C_3$ [13], or $\widehat{b} \cdot f(s) \equiv 0$.

Theorem 2.2. *Let conditions* (2.1)–(2.4), (2.5), (2.9) *be satisfied, and in* (2.3) *we have*

$$p = \begin{cases} 2, & \text{if } n \geq 3, \\ 3, & \text{if } n = 2, \\ 4, & \text{if } n = 1. \end{cases} \tag{2.10}$$

Let also for some number $l > 0$ the following condition be fulfilled

$$\big|q_\varepsilon(t, x, \xi_1) - q_\varepsilon(t, x, \xi_2)\big| \leq l|\xi_1 - \xi_2|. \tag{2.11}$$

Then the limit relation is true

$$J(\overline{y}^\varepsilon, \overline{u}^\varepsilon) \longrightarrow J(\overline{y}, \overline{u}), \ \varepsilon \to 0,$$

where $\{\overline{y}^\varepsilon, \overline{u}^\varepsilon\}$ and $\{\overline{y}, \overline{u}\}$ are optimal processes in problems (1.1)–(1.3) *and* (2.6)–(2.8).

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Dulac–Cherkas Functions for Van Der Pol Equivalent Systems

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The differential equation

$$\frac{d^2x}{dt^2} + \mu(x^2-1)\,\frac{dx}{dt} + x = 0 \tag{1}$$

depending on the real parameter μ has been introduced by the Dutch engineer and physicist Balthasar van der Pol [4] in 1926 to describe self-oscillations in a triod circuit. If we replace t by $-t$ and μ by $-\mu$, then equation (1) remains invariant. Thus, to study the phase portrait of equation (1) we can restrict ourselves to the case $\mu \geq 0$. It is well-known (see, e.g., [3]) that (1) has for $\mu > 0$ a unique limit cycle $\Gamma(\mu)$ which is orbitally stable and hyperbolic. For small μ, the periodic solutions $x = p(t,\mu)$ describing the limit cycle $\Gamma(\mu)$ behave like the solution of the harmonic oscillator, for large μ, $p(t,\mu)$ represents a relaxation oscillation. In what follows we derive differential systems which distinguish in their structure but whose phase portraits are topologically equivalent to that of the van der Pol equation (1). The reason to do this consists in the intension to find the most suitable form for studying the localization and the shape of the limit cycle for arbitrary values of the parameter.

The main tool for our investigation is the method of Dulac–Cherkas functions which was introduced by L. A. Cherkas in 1997 [1] as a generalization of Dulac method [2]. We recall the definition of Dulac–Cherkas function for the planar autonomous differential system

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y) \tag{2}$$

in some open region $\mathcal{G} \subset \mathbb{R}^2$, where $P, Q \in C^1(\mathcal{G}, \mathbb{R})$ and X is the vector field defined by (2).

Definition 1. A function $\Psi \in C^1(\mathcal{G}, \mathbb{R})$ is called the Dulac–Cherkas function of system (2) in $\mathcal{G}$ if there exists a real number $\kappa \neq 0$ such that

$$\Phi(x,y,\kappa) := (\operatorname{grad}\Psi, X) + \kappa\Psi \operatorname{div} X > 0 \ \ (<0) \ \text{ in } \mathcal{G}. \tag{3}$$

In case $\kappa = 1$, Ψ is a Dulac function.

Remark 1. Condition (3) can be relaxed by assuming that Φ may vanish in $\mathcal{G}$ on a set of measure zero, and that no oval of this set is a limit cycle of (2).

For the sequel we introduce the subset $\mathcal{W}$ of $\mathcal{G}$ defined by $\mathcal{W} := \{(x,y) \in \mathcal{G} : \ \Psi(x,y) = 0\}$.

The following theorem can be found in [1,2].

Theorem 1. *Let Ψ be a Dulac–Cherkas function of* (2) *in $\mathcal{G}$. Then any limit cycle Γ of* (2) *located entirely in $\mathcal{G}$ has the following properties:*

(i) Γ *does not intersect $\mathcal{W}$;*

(ii) Γ *is hyperbolic;*

(iii) *the stability of Γ is determined by the sign of the expression $\kappa\Phi\Psi$ on Γ.*

Property (ii) has the strong consequence that the existence of a Dulac–Cherkas function implies that system (2) has no multiple limit cycle.

Theorem 2. *Let Ψ be a Dulac–Cherkas function of* (2) *in $\mathcal{G}$ such that the set $\mathcal{W}$ contains some oval $\mathcal{W}_0$ with the property that the open region $\mathcal{G}_0$ bounded by $\mathcal{W}_0$ belongs to $\mathcal{G}$ and that $\mathcal{G}_0 \cap \mathcal{W}$ is empty. Then there is no limit cycle in $\mathcal{G}_0$.*

Corollary 1. *Under the assumptions of Theorem 2, $\mathcal{W}_0$ can be used as interior boundary of a possible Poincaré–Bendixson annulus.*

The following result is also known [2].

Theorem 3. *Let $\mathcal{G}$ be a simply connected region where Ψ is a Dulac–Cherkas function of* (2) *such that $\mathcal{W}$ consists of one oval in $\mathcal{G}$. Then system* (2) *has at most one limit cycle in $\mathcal{G}$.*

Now we note that (1) can be rewritten as the system

$$\begin{aligned} \frac{dx}{dt} &= -y, \\ \frac{dy}{dt} &= x - \mu(x^2-1)y \end{aligned} \tag{4}$$

of Liénard type. The goal of our investigation is to construct Dulac–Cherkas functions for systems equivalent to the van der Pol system (4) such that the zero-set of these functions consists of a unique oval which can be used by Corollary 1 as interior boundary of a Poincaré–Bendixson annulus containing the unique limit cycle of the corresponding system. At the same time we study the problem whether a Dulac–Cherkas function for an equivalent system can be obtained by applying the equivalence relation to the known Dulac–Cherkas function. We start with the original van der Pol system.

Lemma 1. *The functions*

$$\Psi_a(x,y) := x^2 - 1 + y^2 \tag{5}$$

and

$$\Psi_b(x,y,\mu) := x^2 - \frac{8}{3} + \mu\Big(x - \frac{x^3}{3}\Big)y + y^2 \tag{6}$$

are Dulac–Cherkas functions for system (4) *in $\mathbb{R}^2$ for $\mu > 0$.*

The corresponding expressions (3) read

$$\begin{aligned} \Phi_a(x,-2,\mu) &= 2\mu(x^2-1)^2 \geq 0, \\ \Phi_b(x,-1,\mu) &= \frac{2}{3}\,\mu(x^2-2)^2 \geq 0 \end{aligned}$$

for $\kappa = -2$ and $\kappa = -1$, accordingly.

Remark 2. The zero-sets $\mathcal{W}_a$ and $\mathcal{W}_b(\mu)$ of the functions $\Psi_a(x,y)$ and $\Psi_b(x,y,\mu)$ consist of a unique oval for $\mu > 0$. Thus, these ovals can be used as interior boundaries for a Poincaré-Bendixson annulus. We note that the parameter dependent oval $\mathcal{W}_b(\mu)$ represent for small μ a better approximation of the van der Pol limit cycle $\Gamma(\mu)$.

For the first time the function $\Psi_a(x,y)$ was constructed by L. A. Cherkas in the paper [1]. Next we consider the system

$$\begin{aligned} \frac{d\overline{x}}{dt} &= -\overline{y}, \\ \frac{d\overline{y}}{dt} &= \overline{x} + (\mu - \overline{x}^2)\overline{y}, \end{aligned} \tag{7}$$

which we obtain from system (4) by the scaling $\overline{x} = \sqrt{\mu}\,x, \overline{y} = \sqrt{\mu}\,y$. The representations (4) and (7) are especially useful for small μ: if μ crosses the value 0, the unique limit cycle $\Gamma(\mu)$ in system (4) bifurcates from the family of circles with center at the origin, while in system (7) the limit cycle $\Gamma(\mu)$ bifurcates from the origin (Hopf bifurcation). We note that in the case $\mu = 0$ the phase portraits of these systems are not topologically equivalent.

Lemma 2. *The functions*

$$\overline{\Psi}_a(\overline{x},\overline{y},\mu) := \overline{x}^2 + \overline{y}^2 - \mu \tag{8}$$

and

$$\overline{\Psi}_b(\overline{x},\overline{y},\mu) := \overline{x}^2 + \overline{y}^2 - \frac{8}{3}\,\mu + \Big(\mu\overline{x} - \frac{\overline{x}^3}{3}\Big)\overline{y} \tag{9}$$

are Dulac–Cherkas functions for system (7) *in* $\mathbb{R}^2$ *for* $\mu > 0$.

Both ovals corresponding to the functions (8) and (9) can be used as interior boundaries for a Poincaré-Bendixson annulus. Now we study the singularly perturbed system

$$\begin{aligned} \frac{dx}{d\tau} &= -y, \\ \varepsilon\,\frac{dy}{d\tau} &= x - (x^2 - 1)y, \end{aligned} \tag{10}$$

which we get from system (4) by the scaling $t = \mu\tau$ and using the notation $\varepsilon = 1/\mu^2$. In the case $\mu = 1$ both system coincide such that the functions Ψ_a and Ψ_b defined in (5) and (6) are also Dulac–Cherkas functions of system (7). For $\mu \neq 1$, this scaling changes not only the velocity running along the trajectories but also the vector field such that $\overline{\Psi}_a$ and $\overline{\Psi}_b$ are not longer Dulac–Cherkas functions of system (7).

Lemma 3. *The functions*

$$\Psi_a(x,y,\varepsilon) := x^2 - 1 + \varepsilon y^2$$

and

$$\Psi_b(x,y,\varepsilon) := x^2 - \frac{8}{3} + \Big(x - \frac{x^3}{3}\Big)y + \varepsilon y^2$$

are Dulac–Cherkas functions for system (10) *in* $\mathbb{R}^2$ *for* $\varepsilon > 0$.

In the similar way we derive the following results for three other van der Pol equivalent systems.

Lemma 4. *The function*

$$\Psi(\xi,\eta,\mu) := \xi^2 - \frac{8}{3} + \mu\Big(\eta - \frac{\eta^3}{3}\Big)\xi + \eta^2$$

is a Dulac–Cherkas function for system

$$\begin{aligned}\frac{d\xi}{dt}&=-\eta,\\ \frac{d\eta}{dt}&=\mu\Big(\eta+\frac{\eta^3}{3}\Big)+\xi\end{aligned}$$

in $\mathbb{R}^2$ *for* $\mu>0$.

Lemma 5. *The function*

$$\Psi(\overline{\xi},\overline{\eta},\mu):=\overline{\xi}^2-\frac{8}{3}\,\mu+\Big(\mu\overline{\eta}-\frac{\overline{\eta}^3}{3}\Big)\overline{\xi}$$

is a Dulac–Cherkas function for system

$$\begin{aligned}\frac{d\overline{\xi}}{dt}&=-\overline{\eta},\\ \frac{d\overline{\eta}}{dt}&=\overline{\xi}+\mu\overline{\eta}-\frac{\overline{\eta}^3}{3}\end{aligned}$$

in $\mathbb{R}^2$ *for* $\mu>0$.

Lemma 6. *The function*

$$\Psi(\xi,\eta,\varepsilon):=\xi^2+\varepsilon\eta^2-\frac{8}{3}\,\varepsilon+\Big(\eta-\frac{\eta^3}{3}\Big)\xi$$

is a Dulac–Cherkas function for system

$$\begin{aligned}\frac{d\xi}{d\tau}&=-\eta,\\ \varepsilon\,\frac{d\eta}{d\tau}&=\xi+\eta-\frac{\eta^3}{3}\end{aligned}$$

in $\mathbb{R}^2$ *for* $\varepsilon>0$.

Finally for the van der Pol system we present an approach for the construction of an outer boundary for the Poincaré-Bendixson annulus which does not require an approximation of any orbit.

Theorem 4. *The algebraic ovales*

$$x^2+y^2=1$$

and

$$y^2+\mu yx\Big(2-\frac{x^2}{3}\Big)+(1+\mu^2)x^2-\frac{7\mu^2}{12}\,x^4+\frac{\mu^2}{18}\,x^6-C(\mu)=0$$

form a global algebraic Poincaré-Bendixson annulus for system (4).

In the proof of this theorem we describe a way how the function $C(\mu)$ depending on the parameter μ can be selected. Our approach implies the uniqueness of a limit cycle in the constructed Poincaré-Bendixson annulus.

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Novel Stochastic McKendrick-von Foerster Models with Applications

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The deterministic McKendrick-von Foerster model

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}=-m(t,a)u \ \ (t,a\geq 0) \tag{1}$$

is widely used to examine age-structured populations [2,4,6]. It is usually equipped with the initial condition

$$u(0,a)=\chi(a)\geq 0$$

and the non-local boundary condition

$$u(t,0)=b(t)=\int\limits_0^\infty \beta(t,a)u(t,a)\,da\geq 0.$$

Here $u(t,a)$ is the size (density) of a certain population of a given age $a\geq 0$ at time $t\geq 0$, $m(t,a)\geq 0$ is the per capita mortality rate and $b(t)$ is the birth function that depends on the age-structured size of the population and the per capita birth rate $\beta(t,a)$.

Eq. (1) is a balance equation that can be derived from the basic biophysical principles by letting the increments in time and age be infinitely small and under the assumption that the population is isolated. This explains why the McKendrik-von Foester equation is a source of many specific population models. However, this equation does not take into account stochastic effects, like demographic and environmental fluctuations, which are of importance in any realistic description of population dynamics.

In this presentation, the following stochastic version of this model

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial a}=-(m(t,a)+\dot\nu(t))u \ \ (t,a\geq 0) \tag{2}$$

is considered. Here $\dot\nu(t)$ is a stochastic noise which is represented by the formal (generalized) derivative of a continuous scalar stochastic process $\nu(t)$ defined on the given filtered probability space

$$\big(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbf{P}\big)$$

with the probability measure $\mathbf{P}$ on the σ-algebra $\mathcal{F}$ of subsets of Ω and an increasing sequence of σ-subalgebras $\mathcal{F}_t$ of $\mathcal{F}$, where all the introduced σ-algebras are complete with respect to the measure $\mathbf{P}$.

The aim of the presentation is to deduce the equations for the total size of the juveniles $J(t)$ and the adults $A(t)$

$$J(t) = \int\limits_0^\tau u(t,a)\,da \ \text{ and } \ A(t) = \int\limits_\tau^\infty u(t,a)\,da, \tag{3}$$

where $\tau \geq 0$ is the maturation time [6].

In the assumptions below, the following definition is used.

Definition. A real-valued (deterministic) function $\alpha(t,x,y)$, $t \geq 0$, $x, y \in (-\infty,\infty)$ belongs class L if it is measurable (as a function of three variables) and satisfies the uniform Lipschitz condition with respect to x and y:

$$\big|\alpha(t,x_1,y_1) - \alpha(t,x_2,y_2)\big| \leq L\big(|x_1 - y_1| + |x_2 - y_2|\big)$$

for all $t \geq 0$, $x, y \in (-\infty,\infty)$.

The restrictions on the coefficients in (2) can be summarized as follows:

(A1) The mortality rate $m(t,a)$ is defined as

$$m(t,a) = \begin{cases} m_J(t) := \mu_J(t, J(t), A(t)), & 0 \leq a < \tau, \\ m_A(t) := \mu_A(t, J(t), A(t)), & a \geq \tau, \end{cases} \tag{4}$$

where μ_J and μ_A are class L functions (that is, they are independent of the age a) and $\tau \geq 0$ is the maturation time.

(A2) The function $\chi(a) \geq 0$ (the initial age distribution at time $t = 0$) is càdlàg and satisfies the condition

$$\int\limits_0^\infty \sup_{s \geq a} \chi(s)\,da < \infty.$$

In practical applications, the function χ has a compact support, so that this assumption will be trivially satisfied.

(A3) At any time t, the birth rate function β is defined as

$$\beta(t,a) = \begin{cases} 0, & 0 \leq a < \tau, \\ \beta_A(t) := \beta_A(t, J(t), A(t)), & a \geq \tau, \end{cases}$$

where β_A is a class L function independent of the age a, and by definition, the birth rate of the juvenile population (i.e. β_J) is equal to 0.

(A4) The stochastic process ν is defined as

$$\nu(t) = \int\limits_0^t \gamma(s, J(s), A(s))\,dB(s),$$

where B is the scalar Brownian motion and γ is a class L function.

Below is the main result of the presentation.

Theorem 1. *If assumptions* (A1)–(A4) *are fulfilled, then the aggregated age variables* (3)*, together with the auxiliary variable $X(t)$, satisfy the system*

$$
\begin{aligned}
dJ(t) = {} & \beta_A(t, J(t), A(t))A(t)\,dt - \mu_J(t, J(t), A(t))J(t)\,dt \\
& - \mathcal{D}\big\{t, J(\,\cdot\,), A(\,\cdot\,), X(\,\cdot\,)\big\}\beta_A\big(t-\tau, J(t-\tau), A(t-\tau)\big)A(t-\tau)\,dt \\
& + \gamma(t, A(t), J(t))J(t)dB(t) \ \ (t \geq \tau), \\
dA(t) = {} & -\mu_A(t, J(t), A(t))A(t)\,dt \\
& + \mathcal{D}\big\{t, J(\,\cdot\,), A(\,\cdot\,), X(t)\big\}\beta_A\big(t, J(t-\tau), A(t-\tau)\big)A(t-\tau)\,dt \\
& + \gamma(t, J(t), A(t))A(t)dB(t) \ \ (t \geq \tau), \\
dX(t) = {} & \gamma(t, J(t), A(t))X(t)\,dB(t),
\end{aligned}
\tag{5}
$$

where

$$\mathcal{D}\big\{t, J(\,\cdot\,), A(\,\cdot\,), X(\,\cdot\,)\big\} = \exp\bigg\{-\int\limits_{t-\tau}^{t} \mu_J(s, J(s), A(s))\,ds\bigg\}X(t)X^{-1}(t-\tau)$$

is an integral operator standing for the distributed delay in the equation.

This system satisfies the initial conditions

$$J(t) = J_0(t), \ \ A(t) = A_0(t), \ \ X(t) = X_0(t) \ \ (t \in [0, \tau]),$$

where $J_0(\,\cdot\,)$, $A_0(\,\cdot\,)$ and $X_0(t)$ are $\mathcal{F}_\tau$-measurable, continuous stochastic processes satisfying the following system of stochastic integro-differential equations on the interval $[0, \tau]$:

$$
\begin{aligned}
dJ_0(t) = {} & -\mathcal{D}\big\{t, J_0(\,\cdot\,), A_0(\,\cdot\,)\big\}\,dt + \beta_A(t, J_0(t), A_0(t))A_0(t)\,dt - \mu_J(t, J_0(t), A_0(t))J_0(t)\,dt \\
& + \gamma(t, A_0(t), J_0(t))J_0(t)\,dB(t), \\
dA_0(t) = {} & \mathcal{D}\big\{t, J_0(\,\cdot\,), A_0(\,\cdot\,)\big\}\,dt - \mu_A(t, J_0(t), A_0(t))A_0(t)\,dt \\
& + \gamma(t, J_0(t), A_0(t))A_0(t)\,dB(t), \\
dX_0(t) = {} & \gamma(t, J_0(t), A_0(t))X_0(t)\,dB(t),
\end{aligned}
$$

and

$$\mathcal{D}_0\big\{t, J_0(\,\cdot\,), A_0(\,\cdot\,), X_0(\,\cdot\,)\big\} = \chi(\tau - t)\exp\bigg\{-\int\limits_{0}^{t} \mu_J(s, J_0(s), A_0(s))\,ds\bigg\}X_0(t).$$

The initial conditions for the latter system are given by

$$
\begin{aligned}
J(0) &= \int\limits_0^\tau u(0, s)\,ds = \int\limits_0^\tau \chi(s)\,ds, \\
A(0) &= \int\limits_\tau^\infty u(0, s)\,ds = \int\limits_\tau^\infty \chi(s)\,ds, \\
X(0) &= 1.
\end{aligned}
$$

The proof of this result can be found in [5].

Consider some biologically important stochastic models which can be obtained from Theorem 1.

Example 1: the stochastic counterpart of the recruitment-delayed model.
The following equation is widely used in population dynamics (see, e.g., the monograph [3] or the review paper [1]):

$$A'(t) = \mathcal{B}(A(t-\tau)) - \mathcal{D}(A(t)),$$

where $A(t)$ is the size of the adult population and $\mathcal{B}$ and $\mathcal{D}$ are the birth and death functions, respectively.

Let us deduce a stochastic counterpart of this model starting from the McKendrik-von Foerster equation (2). Assume that

$$m(t,a) = \begin{cases} \mu_J(t), & 0 \le a < \tau, \\ \mu_A(A(t)), & a \ge \tau, \end{cases}$$

and the birth rate is given by

$$\beta(t,a) = \begin{cases} 0, & 0 \le a < \tau, \\ \beta_A(A(t)), & a \ge \tau, \end{cases} \tag{6}$$

where $\beta_A(A)$, $A \in (-\infty, \infty)$, is a continuously differentiable function, which satisfies the assumption $\beta_A'(A) < 0$ for $A \ge 0$.

The coefficient γ in (A4) is a function of t, so that $\nu(t) = \int\limits_0^t \gamma(s)\,dB(s)$, and satisfies the condition $\gamma(t) \ge m > 0$. Then we get

$$dA(t) = -\mu_A(A(t))A(t)\,dt + \alpha(t,\tau)\beta_A(A(t-\tau))A(t-\tau)\,dt + \gamma(t)A(t)\,dB(t),$$

for $t \ge \tau$, where

$$\alpha(t,\tau) = \exp\Big\{ -\int\limits_{t-\tau}^{t} \mu_J(s)\,ds + \nu(t) - \nu(t-\tau) - \frac{1}{2}\int\limits_{t-\tau}^{t} \gamma^2(s)\,ds \Big\}. \tag{7}$$

Example 2: the stochastic counterpart of Nicholson's blowflies model.
The most celebrated model of the deterministic population dynamics is Nicholson's blowflies model and its generalizations (see, e.g., the review paper [1] and the references therein)

$$A'(t) = -m_A A(t) + p_0 A(t-\tau) \exp\big\{ -\theta A(t) \big\}.$$

Consider Eq. (5) for the adult population with the mortality rate $m(t,a)$ and the birth rate $\beta(t,a)$ given by (4) and (6), respectively. Assume also that $\gamma = \gamma(t) \ge m > 0$. Then we get the following stochastic version of the generalized Nicholson's blowflies delay equation:

$$dA(t) = -m_A A(t)\,dt + \alpha(t,\tau)\beta_A(A(t-\tau))A(t-\tau)\,dt + \gamma A(t)A(t)\,dB(t), \tag{8}$$

where $\alpha(t,\tau)$ is given by (7). Notice that this equation differs from that studied in [7], where an additive stochastic noise was appended to the deterministic blowflies model:

$$dA(t) = -m_A A(t)\,dt + p_0 A(t-\tau) \exp\big\{ -\theta A(t-\tau) \big\}\,dt + \delta A(t)\,dB(t). \tag{9}$$

The main difference between Eq. (9), obtained by automatically adding a stochastic noise, and Eq. (8) obtained from the stochastic McKendrik-von Foester model (2) is the presence of the stochastic process $\alpha(t,\tau)$, which represents an intrinsic multiplicative stochastic noise. This random coefficient depends explicitly on the noise $\gamma\dot{B}$, which we added to the mortality rate in (2), and explains how random fluctuations in the population's mortality influence fluctuations in the birth function. This dependence is disregarded in Eq. (9). Note that as long as the noise $\gamma\dot{B}$ is non-zero, we will always get a nontrivial random α in front of the deterministic birth function β_A.

In addition, starting with (2) will always produce a random initial condition $A(t) = \varphi_A(t)$, $0 \le t \le \tau$, as it was shown in the previous section.

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Perron's Effect of Value Change of Characteristic Exponents with a Countable Number of Uniformly Bounded Suslin's Sets

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Consider the linear differential systems

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^2, \ \ t \geq t_0, \tag{1}$$

with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_1(A) \leq \lambda_2(A) < 0$, being linear approximations for the nonlinear systems

$$\dot{y} = A(t)y + f(t, y), \ \ y \in \mathbb{R}^2, \ \ t \geq t_0, \tag{2}$$

likewise with infinitely differentiable so-called m-perturbations $f(t, y)$ of order $m > 1$ of smallness in the neighbourhood of the origin and possible growth outside of it:

$$\|f(t, y)\| \leq C_f \|y\|^m, \ \ y \in \mathbb{R}^2, \ \ t \geq t_0. \tag{3}$$

The known Perron's effect [7], [6, p. 50-51] of value change of characteristic exponents states the existence of systems (1) and (2) with 2-perturbation (3) such that all nontrivial solutions of system (2) turn out to be infinitely continuable and their characteristic exponents take only two values: one is negative, coinciding with the higher exponent $\lambda_2(A) < 0$ of the system of linear approximation (1) and the other one is positive (calculated incidentally in [3, p. 13-15]). Considering this effect as not full (not all nontrivial solutions of the perturbed system (2) take positive exponents), a great number of works were devoted to the investigation of its full version (all nontrivial solutions of system (2) are infinitely continuable to the right and have finite positive exponents (see our last works [4, 5]).

In particular, in these works we have obtained the above-mentioned full Perron's effect corresponding to various types of the collection $\Lambda(A, f) \subset (0, +\infty)$ of Lyapunov's characteristic exponents of all nontrivial solutions of the nonlinear system (2) with m-perturbation (3) for any fixed $m > 1$. In our last works, these bounded collections $\Lambda(A, f) \subset (0, +\infty)$ are completely described by Suslin's sets.

Noteworthy are the results, not connected with Perron's effect: for the exponentially stable systems (2) with perturbations (3) the collections $\Lambda_0(A, f) \subset (-\infty, 0)$ of characteristic exponents of all their nontrivial solutions, emanating from a sufficiently small neighbourhood of the origin, of positive measure are realized in [2], whereas in [1] they are completely described by Suslin's sets.

In investigating the above (not full) Perron's effect, when all nontrivial solutions $y(t, c), y(t_0, c) = c \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ are infinitely continuable and have finite positive as well as negative Lyapunov's exponents $\lambda[y(\,\cdot\,,c)]$, forming respectively non-empty (one-element) sets $\Lambda_+(A,f)$ and $\Lambda_-(A,f)$ and all their collection $\Lambda(A,f)=\Lambda_+(A,f)\cup\Lambda_-(A,f)$, there arises, in particular, the question to what extent they may be common simultaneously (for one system (2)). The answer, as a consequence of a more general result, is obtained in the present report.

Here, for a countable number of uniformly bounded arbitrary Suslin's sets S_k, $k\in\mathbb{N}$ and a partitioning of a plane of initial values of solutions into the same number of domains and segments Π_k we have constructed systems (1) and (2) with m-perturbation (3) such that the characteristic exponents $\lambda[y(\,\cdot\,,c)]$ of nontrivial solutions of system (2) with the initial values $c\in\Pi_k$ make up the sets S_k, and the whole collection of exponents $\Lambda(A,f)$ of nontrivial solutions of that system is the union $\bigcup\limits_{k\in\mathbb{N}} S_k$. The consequence of that common result is the realization of the cases for: a finite number of arbitrary bounded Suslin's sets; two arbitrary, likewise bounded, Suslin's sets

$$S_+\subset(0,+\infty),\quad S_-\subset(-\infty,0),$$

forming the collection $\Lambda(A,f)=S_+\cup S_-$ of characteristic exponents of some system (2).

The following theorem is valid.

Theorem 1. *For any parameters $m>1$, $\lambda_1\le\lambda_2<0$ and any two sequences S_{in}, $n\in\mathbb{N}$, $i=1,2$, uniformly bounded by Suslin's sets*

$$S_{1n}\subset\big[\lambda_1+\varepsilon,b_1\big],\quad S_{2n}\subset\big[\max\{\lambda_2+\varepsilon,b_1\},b_2\big],\ \ n\in\mathbb{N},$$

with number $\varepsilon>0$, which are the sets of values respectively of the functions $\beta_{1n}(\,\cdot\,)$ and $\beta_{2n}(\,\cdot\,)$ of the 1st Baire class on every of the half-intervals $(n-1,n]$ and $[-n,-n+1)$, there exist:

1) *a system of linear approximation (1) with bounded infinitely differentiable coefficients and characteristic exponents $\lambda_i(A)=\lambda_i$, $i=1,2$;*

2) *an infinitely differentiable m-perturbation $f(t,y)$ such that all nontrivial solutions $y(t,c)$ with the initial conditions*
$$y(t_0,c)=(c_1,c_2)\in\mathbb{R}^2\setminus\{\mathbf{0}\}$$
of the perturbed system (2) are infinitely continuable to the right, and their characteristic exponents $\lambda[y(\,\cdot\,,c)]$ form for every $n\in\mathbb{N}$ the collections
$$\Big\{\lambda\big[y(\,\cdot\,,(c_1,0))\big]:\ |c_1|\in(n-1,n]\Big\}=S_{1n},$$
$$\Big\{\lambda\big[y(\,\cdot\,,c)\big]:\ |c_2|\in(n-1,n]\Big\}=S_{2n},\quad \Lambda(A,f)=\bigcup_{i,n}S_{in},$$
on every of the above-mentioned intervals separately.

The theorem below gives us the answer to the question posed at the beginning of our report.

Theorem 2. *For any parameters $m>1$, $\lambda_1\le\lambda_2<0$ and arbitrary bounded Suslin's sets $S_-\subset(-\infty,0)$ and $S_+\subset(0,+\infty)$ there exist the nonlinear system (2) with the linear approximation (1), having characteristic exponents $\lambda_i(A)=\lambda_i$, $i=1,2$, and m-perturbation (3) such that all nontrivial solutions $y(t,c)$ are infinitely continuable and their Lyapunov's exponents $\lambda[y(\,\cdot\,,c)]$ form the sets*

$$\big\{\lambda[y(\,\cdot\,,c)]:\ c=(c_1,0)\neq0\big\}=S_-,\quad \big\{\lambda[y(\,\cdot\,,c)]:\ c_2\neq0\big\}=S_+.$$

This theorem is a direct consequence of Theorem 1 and its proof.

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On One Nonlinear Degenerate Integro-Differential Equation of Parabolic Type

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A lot of scientific works are dedicated to the investigation and numerical resolution of integro-differential models (see, for example, [2, 7, 9, 11, 16] and the references therein).

One type of nonlinear integro-differential parabolic model is obtained at the mathematical simulation of processes of electromagnetic field penetration into a substance. Based on Maxwell system [12], the mentioned model at first appeared in [3]. The integro-differential system obtained in [3] describes many other processes as well (see, for example, [7, 9] and the references therein). Equations and systems of such types still yield to the investigation for special cases. In this direction the latest and rather complete bibliography can be found in the following monographs [7, 9].

The purpose of this note is to analyze degenerate one-dimensional case of such type equations. Unique solvability and convergence of the constructed semi-discrete scheme with respect to the spatial derivative and fully discrete finite difference scheme are studied.

The investigated problem has the following form. In the rectangle $Q = (0, 1) \times (0, T]$, where T is a fixed positive constant, we consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left\{\left[\int\limits_0^t \Big(\frac{\partial U}{\partial x}\Big)^2 \, d\tau + \Big(\frac{\partial U}{\partial x}\Big)^2\right] \frac{\partial U}{\partial x}\right\} = f(x,t), \tag{1}$$

$$U(0,t) = U(1,t) = 0, \quad t \in [0,T], \tag{2}$$

$$U(x,0) = U_0(x), \quad x \in [0,1]. \tag{3}$$

Here $f = f(x,t)$, $U_0 = U_0(x)$ are given functions of their arguments and $U = U(x,t)$ is an unknown function. It is necessary to mention that (1) is a degenerate type parabolic equation with integro-differential and p-Laplacian ($p = 4$) terms. Let us note that for non-degenerate variants of (1)–(3) type problem for more general nonlinearities are studied in [4]. Many works are devoted to the investigation of multi-dimensional cases of such type equations and systems as well (see, for example, [1, 5, 7–10, 13] and the references therein). We would also like to note that in recent years special attention has been paid to the construction and investigation of splitting models for this type and their generalized variants of multi-dimensional integro-differential equations (see, for example, [7, 8] and the references therein).

As it was already mentioned, (1) type models arise, on the one hand, when solving real applied problems, and on the other hand, as a natural generalization of some nonlinear parabolic equations and systems studied, for example, in [14, 15] and in many other works as well.

Problems of (1)–(3) type at first were studied in [10], where the monotonicity of the considered operator is proved and the unique solvability is obtained.

Applying one modification of compactness method developed in [15] (see also [14]) the following uniqueness and existence statement takes place [5].

Theorem 1. *If $f \in W_2^1(Q)$, $f(x,0)=0$, $U_0 \in \overset{\circ}{W}{}_2^1(0,1)$, then there exists the unique solution U of problem* (1)–(3) *satisfying the following properties:*

$$U \in L_4\big(0,T;\overset{\circ}{W}{}_4^1(0,1)\cap W_2^2(0,1)\big), \ \ \partial U/\partial t \in L_2(Q), \ \ \sqrt{T-t}\,\partial^2 U/\partial t\partial x \in L_2(Q).$$

Here usual well-known spaces are used.

In order to describe the space-discretization for problem (1)–(3), let us introduce nets: $\omega_h = \{x_i = ih,\ i=1,2,\dots,M-1\}$, $\overline{\omega}_h = \{x_i = ih,\ i=0,1,\dots,M\}$ with $h=1/M$. The boundaries are specified by $i=0$ and $i=M$. The semi-discrete approximation at (x_i,t) is designed by $u_i=u_i(t)$. The exact solution of problem (1)–(3) at point (x_i,t) is denoted by $U_i=U_i(t)$.

Approximating the space derivatives by a forward and backward differences

$$u_{x,i} = \frac{u_{i+1}-u_i}{h}\,, \quad u_{\overline{x},i} = \frac{u_i-u_{i-1}}{h}\,,$$

let us correspond the following semi-discrete scheme to problem (1)–(3):

$$\frac{du_i}{dt} - \bigg\{\bigg[\int\limits_0^t (u_{\overline{x},i})^2\,d\tau + (u_{\overline{x},i})^2\bigg]u_{\overline{x},i}\bigg\}_{x,i} = f(x_i,t), \ \ i=1,\dots,M-1, \tag{4}$$

$$u_0(t) = u_M(t) = 0, \ \ t\in[0,T], \tag{5}$$

$$u_i(0) = U_{0,i}, \ \ i=0,1,\dots,M, \tag{6}$$

which approximates problem (1)–(3) on smooth solutions with the first order of accuracy with respect to spatial step h.

The semi-discrete scheme (4)–(6) represent a Cauchy problem for nonlinear system of ordinary integro-differential equations. It is stable with respect to initial data and right-hand side of equation (4) in the norm

$$\|u\|_h = (u,u)_h^{1/2}, \quad (u,v)_h = \sum_{i=1}^{M-1} u_i v_i h.$$

It is not difficult to obtain following estimate for (4)–(6):

$$\|u\|_h^2 + \int\limits_0^t ||u_{\overline{x}}]|_h^2\,d\tau < C,$$

where the norm under integral is

$$||u]|_h^2 = (u,u]_h = \sum_{i=1}^{M} u_i u_i h.$$

Here C denotes the positive constant independent of the mesh parameter h. This estimate gives the above-mentioned stability as well as the global existence of a solution to problem (4)–(6).

Here in Theorem 2 and below in Theorem 3, using an approach of the work [6] for investigation of finite-difference scheme, the convergence of the approximate solutions are proved.

The following statement takes place.

Theorem 2. *The solution $u(t) = (u_1(t),u_2(t),\dots,u_{M-1}(t))$ of the semi-discrete scheme* (4)–(6) *converges to the solution $U(t) = (U_1(t),U_2(t),\dots,U_{M-1}(t))$ of problem* (1)–(3) *in the norm $\|\cdot\|_h$ as $h\to 0$.*

In order to describe the fully discrete analog of problem (1)–(3), let us construct grid on the rectangle $\overline{Q}$. For using the time-discretization in equation (1), the net is introduced as follows $\omega_\tau = \{t_j = j\tau,\ j = 0, 1, \dots, J\}$, with $\tau = T/J$ and $\overline{\omega}_{h\tau} = \overline{\omega}_h \times \omega_\tau$, $u_i^j = u(x_i, t_j)$.

Let us correspond the following implicit finite difference scheme to problem (1)–(3)), where the term with time derivative in (4) is approximated using the forward finite difference formula:

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \bigg\{\Big[\tau \sum_{k=1}^{j+1} (u_i^k)^2 + (u_{\overline{x},i}^{j+1})^2\Big] u_{\overline{x},i}^{j+1}\bigg\}_{x,i} = f_i^{j+1}, \quad i = 1, 2, \dots, M-1, \quad j = 0, 1, \dots, J-1; \tag{7}$$

$$u_0^j = u_M^j = 0, \quad j = 0, 1, \dots, J, \tag{8}$$

$$u_i^0 = U_{0,i}, \quad i = 0, 1, \dots, M. \tag{9}$$

Thus, the system of nonlinear algebraic equations (7)–(9) is obtained, which approximates problem (1)–(3) on sufficiently smooth solution with the first order of accuracy with respect to time and spatial steps τ and h.

The following estimate can be obtained easily for the finite difference scheme (7)–(9):

$$\max_{0 \le j\tau \le T} \|u^j\|_h^2 + \sum_{k=1}^{J} ||u_{\overline{x}}^k]|_h^2 \tau < C,$$

which guarantees the stability and solvability of the scheme (7)–(9). It is proved also that system (7)–(9) has a unique solution.

Here C represents positive constant independent from time and spatial steps τ and h.

The following main conclusion is valid for scheme (7)–(9).

Theorem 3. *The solution $u^j = (u_1^j, u_2^j, \dots, u_{M-1}^j)$, $j = 1, 2, \dots, J$ of the difference scheme* (7)–(9) *converges to the solution $U^j = (U_1^j, U_2^j, \dots, U_{M-1}^j)$, $j = 1, 2, \dots, J$ of problem* (1)–(3) *in the norm $\|\cdot\|_h$ as $\tau \to 0$ and $h \to 0$.*

Note that for solving the difference scheme (7)–(9) the Newton iterative process is used. Various numerical experiments are done. These experiments agree with theoretical research.

It is very interesting to look for assumptions on the data of the considered problem (1)–(3) that provide the regularity for the solution $U(x,t)$, which is required for obtaining rates of convergence in Theorems 2 and 3 as well as the optimal rates of convergence. It is important also to study more general nonlinearities for such kind degenerate and non-degenerate models.

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Structure of Nonoscillatory Solutions of Second Order Half-Linear Differential Equations

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We consider second order half-linear differential equations of the form

$$(p(t)\varphi_\alpha(x'))' + q(t)\varphi_\alpha(x) = 0, \tag{A}$$

where $\alpha > 0$ is a constant, $p(t)$ and $q(t)$ are positive continuous functions on $[a,\infty)$, $a \geqq 0$, and $\varphi_\alpha(u)$ is an odd function on $\mathbb{R}$ defined by

$$\varphi_\alpha(u) = |u|^{\alpha-1}u = |u|^\alpha \operatorname{sgn} u, \ \ u \in \mathbb{R}.$$

It is known that all proper solutions of (A) are oscillatory, or else nonoscillatory. Equation (A) itself is said to be oscillatory (or nonoscillatory) if all of its proper solutions are oscillatory (or nonoscillatory). We are concerned exclusively with the nonoscillatory equation (A) with $p(t)$ and $q(t)$ satisfying the conditions

$$\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt = \infty \ \text{ and } \ \int\limits_a^\infty q(t)\,dt < \infty. \tag{1}$$

Extensive use is made of the functions $P_\alpha(t)$ and $\rho(t)$ defined by

$$P_\alpha(t) = \int\limits_a^t p(s)^{-\frac{1}{\alpha}}\,ds, \quad \rho(t) = \int\limits_t^\infty q(s)\,ds.$$

The purpose of this paper is to report to the QUALITDE – 2020 a result that has recently been obtained in our efforts to gain precise information about the overall structure of solutions of nonoscillatory equations of the form (A).

We begin by noting that a simple criterion for nonoscillation of (A) is given by

$$P_\alpha(t)\rho(t)^{\frac{1}{\alpha}} \leqq \frac{\alpha}{1+\alpha} \ \text{ for all large } t.$$

Putting $D_\alpha x(t) = p(t)\varphi_\alpha(x'(t))$, we call it the quasi-derivative of $x(t)$. Let $x(t)$ be a nonocillatory solution of (A). Because of (1) both $x(t)$ and $D_\alpha x(t)$ are of the same sign and have the limits $x(\infty) = \lim\limits_{t\to\infty} x(t)$ and $D_\alpha x(\infty) = \lim\limits_{t\to\infty} D_\alpha x(t)$ in the extended real number system. There are three patterns of the pair $\{x(\infty), D_\alpha x(\infty)\}$, namely,

(i) $|x(\infty)| = \infty$, $0 < |D_\alpha x(\infty)| < \infty$;

(ii) $|x(\infty)| = \infty$, $D_\alpha x(\infty) = 0$;

(iii) $0 < |x(\infty)| < \infty$, $D_\alpha x(\infty) = 0$.

A solution of (A) satisfying (i), (ii) or (iii) are named, respectively, a *maximal*, *minimal*, or *intermediate solution*. If $x(t)$ is a nonoscillatory solution of (A), then the functions $u(t)$, $v(t)$ defined by

$$u(t) = \frac{D_\alpha x(t)}{\varphi_\alpha(x(t))}, \quad v(t) = -\frac{x(t)}{\varphi_{\frac{1}{\alpha}}(D_\alpha x(t))}$$

satisfy the first order differential equations

$$u' = -q(t) - \alpha p(t)^{-\frac{1}{\alpha}} |u|^{1+\frac{1}{\alpha}}, \tag{R1}$$

$$v' = -p(t)^{-\frac{1}{\alpha}} - \frac{1}{\alpha} q(t)|v|^{1+\alpha}. \tag{R2}$$

Conversely, it is shown that if (R1) or (R2) has a global solution, then (A) possesses a nonoscillatory solution. Our study was motivated by the ambitious conjecture that all nonoscillatory solutions can be reproduced from appropriate global solutions of (R1) or (R2). Equations (R1) and (R2) are referred to as the generalized Riccati differential equations (Riccati equations for short) associated with equation (A).

It turns out that the existence of these three types of solutions of (A) essentially depends on the convergence or divergence of the integrals

$$I = \int\limits_a^\infty p(t)^{-\frac{1}{\alpha}} \rho(t)^{\frac{1}{\alpha}}\, dt \ \text{ and } \ J = \int\limits_a^\infty q(t) P_\alpha(t)^\alpha\, dt. \tag{2}$$

Let us distinguish the following four cases:

$$\text{(i)}\ \ I < \infty \wedge J < \infty; \ \ \text{(ii)}\ \ I = \infty \wedge J < \infty; \ \ \text{(iii)}\ \ I < \infty \wedge J = \infty; \ \ \text{(iv)}\ \ I = \infty \wedge J = \infty. \tag{3}$$

Notice that (ii) holds only if $\alpha > 1$ and that (iii) holds only if $\alpha < 1$.

Analysis of the first three cases in (3) can be made without difficulty and leads to the following expected result.

Theorem 1. *If* (3)-(i) *holds, then* (A) *possesses a maximal solution and a minimal solution.*

Theorem 2. *If* (3)-(ii) *holds, then* (A) *possesses a maximal solution and an intermediate solution.*

Theorem 3. *If* (3)-(iii) *holds, then* (A) *possesses a minimal solution and an intermediate solution.*

What is anticipated for the case (3)-(iv) is the existence of at least one intermediate solution of (A). This, however, seems to be difficult to prove, and for now we have to be content with giving a less general result on the basis of the inequality

$$\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}} \rho(t)^{1+\frac{1}{\alpha}}\, dt < \infty,$$

which is a necessary condition for nonoscillation of (A).

Theorem 4. *Let* (3)-(iv) *hold.* (A) *possesses an intermediate solution if*

$$\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\rho(t)^{1+\frac{1}{\alpha}}\,ds \leqq (1+\alpha)^{-1-\frac{1}{\alpha}}\rho(t) \ \text{ for all large } t. \tag{4}$$

Taking into account the duality between the integrals I and J in (2), one can formulate the following theorem which is also true.

Theorem 5. *Let* (3)-(iv) *hold.* (A) *possesses an intermediate solution if*

$$\int\limits_a^t q(s)P_\alpha(s)^{1+\alpha}\,ds \leqq \Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}P_\alpha(t) \ \text{ for all large } t.$$

Example. Let (A_0) denote a special case of (A) with $q(t)$ given by

$$q(t)=\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}p(t)^{-\frac{1}{\alpha}}P_\alpha(t)^{-1-\alpha}.$$

It is clear that $p(t)$ and $q(t)$ satisfy (3)-(iv) and both Theorems 4 and 5 are applicable to (A_0). Notice that (A_0) has an exact intermediate solution $x_0(t)=P_\alpha(t)^{\frac{\alpha}{1+\alpha}}$.

The feature of our work is that all the solutions of (A) mentioned in the above theorems are reproduced from appropriate global solutions of the Riccati equations (R1) and (R2) whose existence is established by means of fixed point techniques. Such a systematic attempt at reproduction of nonoscillatory solutions of (A) from global solutions of the associated Riccati equations was undertaken for the first time by the present authors [2]. The merit of our approach is that the solutions sought can be represented as explicit exponential–integral formulas in terms of global solutions of (R1) or (R2).

It should be emphasized that some of the results presented here are already known (see, e.g., [1]), but our purpose is to show that an entirely different approach can be used to develop a systematic existence theory of nonoscillatory solutions for second order half-linear differential equations.

Remark. Needless to say, entirely parallel results can also be obtained for the nonoscillatory equation (A) with $p(t)$ and $q(t)$ satisfying

$$\int\limits_a^\infty p(t)^{-\frac{1}{\alpha}}\,dt<\infty \ \text{ and } \ \int\limits_a^\infty q(t)\,dt=\infty. \tag{5}$$

Our article devoted to the study of two types of nonoscillation for equation (A) satisfying (1) and (5) combined will be published in the near future.

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On Exact Solutions of Karman's Equation in Nonlinear Theory of Gas Dynamics

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In the plane of independent variables x and y consider quasilinear Karman's equation, arising in a variety of physical problems such as nonlinear vibrations, and irrotational transonic flows of baritropic gas [1–4, 6, 12],

$$(u_x)^\alpha u_{xx} - u_{yy} = 0. \tag{1}$$

Equation (1) is considered in the class of hyperbolic solutions which in this case is determined by the condition

$$u_x > 0. \tag{2}$$

Let

$$m := \frac{\alpha}{2(\alpha+2)}\,, \quad -2 \neq \alpha \in \mathbb{R} := (-\infty, +\infty). \tag{3}$$

Theorem. *If the condition $m \in \mathbb{N} := \{1,2,3,\dots\}$ is fulfilled, then the general classical solution $u \in C^2$ of equation (1) is given by the formulas*

$$\begin{cases} x = (X-Y)^{2m+1}\,\dfrac{\partial^{2m}}{\partial X^m \partial Y^m}\,\dfrac{F(X)-G(Y)}{X-Y}\,, \\[2mm] y = m[2(1-2m)]^{2m}\,\dfrac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}}\,\dfrac{F'(X)-G'(Y)}{X-Y}\,, \\[2mm] u = m[2(1-2m)]^{2m}\Big[\Big(\dfrac{m-1}{2m-1}\,X + \dfrac{m}{2m-1}\,Y\Big)\,\dfrac{\partial^{2m-2}}{\partial X^{m-1}\partial Y^{m-1}}\,\dfrac{F'(X)-G'(Y)}{X-Y} \\[2mm] \qquad\qquad -\dfrac{m-1}{2m-1}\,\dfrac{\partial^{2m-3}}{\partial X^{m-2}\partial Y^{m-1}}\,\dfrac{F'(X)-G'(Y)}{X-Y}\Big] \ \ \textit{for}\ \ m = 2,3,\dots \end{cases} \tag{4}$$

and

$$\begin{cases} x = -2[F(X)-G(Y)] + [F'(X)+G'(Y)](X-Y), \\[2mm] y = \dfrac{4[F'(X)-G'(Y)]}{X-Y}\,, \\[2mm] u = \dfrac{4[YF'(X)-XG'(Y)]}{X-Y} \ \ \textit{for}\ \ m=1. \end{cases} \tag{5}$$

Here $F,G \in C^{m+1}$ are arbitrary functions with respect to the variables X and Y, respectively.

Proof. Let us introduce the Riemann invariants of equation (1) as independent variables

$$\begin{cases} X = q + \dfrac{2}{\alpha+2}\,p^{\frac{\alpha+2}{2}}, \\[2mm] Y = q - \dfrac{2}{\alpha+2}\,p^{\frac{\alpha+2}{2}}, \end{cases} \tag{6}$$

in terms of which equation (1) can be rewritten in the form of a system of equations of the first order [7,8]

$$\begin{cases} X_y - p^{\frac{\alpha}{2}} X_x = 0, \\ Y_y + p^{\frac{\alpha}{2}} Y_x = 0. \end{cases} \tag{7}$$

Here $p := u_x$, $q := u_y$.

In system (7), we choose X and Y as the independent variables, while $x(X,Y)$ and $y(X,Y)$ as the desired functions. Applying the formulas of differentiation of implicit functions of two variables

$$x_X = DY_y, \quad x_Y = -DX_y, \quad y_X = -DY_x, \quad y_Y = DX_x,$$

where $D := \frac{D(x,y)}{D(X,Y)}$ is the Jacobian of transformation, from system (7) we obtain

$$\begin{cases} x_X - p^{\frac{\alpha}{2}} y_X = 0, \\ x_Y + p^{\frac{\alpha}{2}} y_Y = 0. \end{cases} \tag{8}$$

Here $p^{\frac{\alpha}{2}} = \big\{\frac{1}{2(1-2m)}(X-Y)\big\}^{2m}$ due (2), (3) and (6).

Eliminating the function $y(X,Y)$ from system (8) we obtain that the function $x(X,Y)$ satisfies the Euler–Poisson–Darboux–Riemann equation [4,10]

$$x_{XY} + \frac{m}{X-Y}\, x_X - \frac{m}{X-Y}\, x_Y = 0. \tag{9}$$

By a similar way for the function $y(X,Y)$ we get

$$y_{XY} - \frac{m}{X-Y}\, y_X + \frac{m}{X-Y}\, y_Y = 0. \tag{10}$$

General solutions of equations (9) and (10) under the conditions of the theorem have the following form [9,11]

$$\begin{cases} x = (X-Y)^{2m+1} \dfrac{\partial^{2m}}{\partial X^m \partial Y^m} \dfrac{F_1(X) - G_1(Y)}{X-Y}, \\ y = \dfrac{\partial^{2m-2}}{\partial X^{m-1} \partial Y^{m-1}} \dfrac{F_2(X) - G_2(Y)}{X-Y}\,, \end{cases} \tag{11}$$

respectively. Here $F_1, G_1 \in C^{m+2}$ and $F_2, G_2 \in C^{m+1}$ are arbitrary functions.

Taking into account (11), satisfying system (8), we get

$$F_2(X) = m[2(1-2m)]^{2m} F_1'(X), \quad G_2(Y) = m[2(1-2m)]^{2m} G_1'(Y). \tag{12}$$

Further, to obtain the final form of the function u, due (3), (6) and (8) we have

$$\begin{aligned} du &= p\,dx + q\,dy \\ &= p(x_X\,dX + x_Y\,dY) + q(y_X\,dX + y_Y\,dY) = (q + p^{\frac{\alpha+2}{2}}) y_X\,dX + (q - p^{\frac{\alpha+2}{2}}) y_Y\,dY \\ &= \Big(\frac{m-1}{2m-1}\,X + \frac{m}{2m-1}\,Y\Big) y_X\,dX + \Big(\frac{m}{2m-1}\,X + \frac{m-1}{2m-1}\,Y\Big) y_Y\,dY, \end{aligned}$$

whence

$$U_X = \Big(\frac{m-1}{2m-1}\,X + \frac{m}{2m-1}\,Y\Big) y_X, \quad U_Y = \Big(\frac{m}{2m-1}\,X + \frac{m-1}{2m-1}\,Y\Big) y_Y. \tag{13}$$

By virtue of the first equality in (13), we obtain

$$\begin{aligned} U(X,Y) &= \frac{m-1}{2m-1} \int X y_X \, dX + \frac{m}{2m-1}\, Yy + \varphi(Y) \\ &= \frac{m-1}{2m-1} \Big(Xy - \int y \, dX \Big) + \frac{m}{2m-1}\, Yy + \varphi(Y), \end{aligned} \tag{14}$$

where φ is an arbitrary function.

According to the second equality from (13), for definition of the function φ, we get

$$\frac{m-1}{2m-1} \Big(X y_Y - \int y_Y \, dX \Big) + \frac{m}{2m-1}\, (y + Y y_Y) + \varphi'(Y) = \Big(\frac{m}{2m-1}\, X + \frac{m-1}{2m-1}\, Y \Big) y_Y. \tag{15}$$

By virtue of (10), we obtain

$$\int y_Y \, dX = \int \Big(\frac{Y-X}{m}\, y_{XY} + y_X \Big) \, dX = \frac{Y-X}{m}\, y_Y + \frac{1}{m} \int y_Y \, dX + y.$$

Thus, we have

$$\int y_Y \, dX = \frac{Y-X}{m-1}\, y_Y + \frac{m}{m-1}\, y \ \text{ for } \ m \neq 1.$$

Taking into account the latter equality, from (15) we obtain

$$\varphi'(Y) \equiv 0 \implies \varphi = const \ \text{ for } \ m = 2, 3, \dots . \tag{16}$$

Analogously, from (14) for $m = 1$, we get

$$U(X,Y) = Yy + \varphi(Y). \tag{17}$$

According to the second equality from (13) for $m = 1$, for definition of the function φ, we get

$$\varphi'(Y) = (X - Y) y_Y - y = -G_2'(Y) \implies \varphi(Y) = -G_2(Y). \tag{18}$$

Now, introducing the notation $F := F_1$, $G := G_1$ and taking into account (11), (12), (14), (16)–(18), we obtain (4) and (5), respectively. □

Remark. In the case $m = 1$, i.e. for $\alpha = -4$, the solution (5) of equation (1) by the method of Lee's group has been obtained in [5].

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Asgeirsson Principle and Exact Boundary Controllability Problems for One Class of Hyperbolic Systems

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In the domain $D_T: \ 0<x<l, \ 0<t<T$, of the plane O_{xt} of independent variables x, t consider a hyperbolic system of the following form

$$u_{tt}-Au_{xx}=F(x,t), \ \ (x,t)\in D_T, \tag{1}$$

where A is a symmetric positively defined constant square matrix of order n, $F=(F_1(x,t),\dots,F_n(x,t))$ is given and $u=(u_1(x,t),\dots,u_n(x,t))$ – unknown vector-functions, $n\geq 2$.

For system (1) consider an initial-boundary problem with the following statement: in the domain D_T find a solution $u=u(x,t)$ to system (1) that satisfies the following initial conditions

$$u(x,0)=\varphi(x), \quad u_t(x,0)=\psi(x), \ \ 0<x<l, \tag{2}$$

and the boundary conditions

$$u(0,t)=\mu_1(t), \quad u(l,t)=\mu_2(t), \ \ 0<t<T, \tag{3}$$

where

$$\varphi=(\varphi_1(x),\dots,\varphi_n(x)), \ \ \psi=(\psi_1(x),\dots,\psi_n(x)), \ \ \mu_i(t)=(\mu_{i1}(t),\dots,\mu_{in}(t)), \ \ i=1,2,$$

are given vector-functions.

As is known, problem (1), (2), (3) is posed correctly. We consider generalized solutions u of this problem in the space $C_0(\overline{D}_T)$ in the sense of the theory of distribution. Here the space $C_0(\overline{D}_T)$ is obtained by completion of the space $C^1(\overline{D}_T)$ with respect to the norm

$$\|u\|_{C_0(\overline{D}_T)}=\|u\|_{C(\overline{D}_T)}+\|u_t(x,0)\|_{C([0,l])},$$

and consists of continuous vector-functions u from $\overline{D}_T$ having continuous classical derivative u_t for $t=0$, $x\in[0,l]$. In this case, from the data of problem (1), (2), (3), i.e. from φ, ψ, μ_1, μ_2 and F, we require that

$$\varphi\in C([0,l]), \ \ \psi\in C([0,l]), \ \ \mu_i\in C_0([0,T]), \ \ i=1,2; \ \ F\in C(\overline{D}_T), \tag{4}$$

and at the points $O(0,0)$ and $O_1(0,0)$ there are valid the following necessary conditions of agreement

$$\mu_1(0)=\psi(0), \ \ \mu_2(0)=\varphi(l), \ \ \mu_1'(0)=\psi(0), \ \ \mu_2'(0)=\psi(l), \tag{5}$$

where the space $C_0([0,T])$ is obtained by completion of the space $C^1([0,T])$ with respect to the norm

$$\|\mu\|_{C_0([0,T])}=\|\mu\|_{C([0,T])}+\|\mu'(0)\|$$

and consists of the traces of vector-functions from the space $C_0(\overline{D}_T)$ on the side $\{x = 0,\ 0 \le t \le T\}$ of the rectangle D_T. At fulfillment of conditions (4), (5), problem (1), (2), (3) has a unique solution u in the space $C_0(\overline{D}_T)$. This solution will be a classical solution in the space $C^2(\overline{D}_T)$ if instead of (4) we require that

$$\varphi \in C^2([0,l]),\ \ \psi \in C^1([0,l]),\ \ \mu_i \in C^2([0,T]),\ \ i = 1,2;\ \ F \in C^1(\overline{D}_T),$$

besides, in this case, at the points $O(0,0)$ and $O_1(0,0)$, together with (5) should be additionally fulfilled the following conditions of agreement

$$\mu_1''(0) - A\varphi''(0) = F(0,0),\ \ \mu_2''(0) - A\varphi''(l) = F(l,0).$$

Problem (1), (2), (3) is said to be controllable, if for "arbitrary" initial data φ, ψ and the right-hand side F of system (1), there exist appropriate "control" vector-functions μ_1 and μ_2 such that the solution of problem (1), (2), (3) satisfies the conditions

$$u(x,T) = u_t(x,T) = 0,\ x \in [0,l]. \tag{6}$$

Denote by k_i the characteristic numbers of the matrix A, and by v_i – the corresponding eigenvectors, i.e. $Av_i = k_i v_i$. $i = 1,\dots,n$. According to our requirements imposed on the matrix A we have

$$k_i = \lambda_i^2,\ \ \lambda_i = const > 0,\ \ i = 1,\dots,n. \tag{7}$$

Due to (7) the hyperbolic system (1) has the following families of characteristic lines

$$x + \lambda_i t = const\ \text{ and }\ x - \lambda_i t = const,\ \ i = 1,\dots,n.$$

Denote by K a square matrix of order n whose columns are vectors $v_1,\dots,v_n$. It is obvious that $\det K \ne 0$ and denote by $w_1,\dots,w_n$ the components of the vector $K^{-1}u$ where u is a solution of system (1).

Denote by $P_0^iP_1^iP_2^iP_3^i$ the characteristic parallelogram whose sides $P_0^iP_1^i$ and $P_2^iP_3^i$ belong to the family of characteristic lines $x - \lambda_i t = const$, while sides $P_0^iP_2^i$ and $P_1^iP_3^i$ belong to the characteristic lines $x + \lambda_i t = const$, besides, the coordinate of point P_0^i with respect to the variable t exceeds the coordinates of the rest points P_1^i, P_2^i and P_3^i with respect to the same variable, $i = 1,\dots,n$.

Generalized Asgeirsson principle: for the components $w_1,\dots,w_n$ of the vector $K^{-1}u$, where $u \in C_0(\overline{D}_T)$ is a generalized solution of system (1), the following equalities

$$w_i(P_0^i) = w_i(P_1^i) + w_i(P_2^i) - w_i(P_3^i) + \frac{1}{2\lambda_i} \int\limits_{P_0^iP_1^iP_2^iP_3^i} K^{-1}F(x,t)\,dx\,dt,\ \ i = 1,\dots,n,$$

are valid, where $P_0^iP_1^iP_2^iP_3^i$ is an arbitrary characteristic parallelogram lying in $\overline{D}_T$.

Below, for simplicity of presentation we will assume that $F = 0$.

Remark 1. If $T = \frac{l}{\lambda_i}$, $1 \le i \le n$, then for existence of the solution $u = u(x,t) \in C_0(\overline{D}_T)$ of problem (1), (2), (3), satisfying condition (6) it is necessary that the data of this problem φ and ψ satisfy the following condition

$$\widetilde{\varphi}_i(0) + \widetilde{\varphi}_i(l) + \frac{1}{\lambda_i} \int\limits_0^l \widetilde{\psi}_i(\xi)\,d\xi = 0, \tag{8}$$

where

$$\widetilde{\varphi} = (\widetilde{\varphi}_1,\dots,\widetilde{\varphi}_n) = K^{-1}\varphi,\ \ \widetilde{\psi} = (\widetilde{\psi}_1,\dots,\widetilde{\psi}_n) = K^{-1}\psi.$$

The proof of the following theorem is based on the generalized Asgeirsson principle.

Theorem. *Let* $T \geq T_0 = \max\limits_{1 \leq i \leq n} \frac{l}{\lambda_i}$ *and the vector-functions* $\varphi \in C([0,l])$, $\psi \in C([0,l])$ *be given which satisfy conditions* (8) *for* $i = 1, \dots, n$*. Then there exist vector-functions* $\mu_1, \mu_2 \in C_0([0,T])$ *satisfying the condition of agreement* (5) *such that the solution* $u \in C_0(\overline{D}_T)$ *of problem* (1), (2), (3) *satisfies condition* (6)*.*

Remark 2. If $T < T_0 = \max\limits_{1 \leq i \leq n} \frac{l}{\lambda_i}$, then not for all $\varphi \in C([0,l])$, $\psi \in C([0,l])$ problem (1), (2), (3) is exactly controllable.

Remark 3. At fulfillment of the conditions of the above theorem, uniqueness of the vector-functions μ_1 and μ_2 will hold when $\lambda_0 := \lambda_1 = \lambda_2 = \cdots = \lambda_n$ for $T = T_0 = \frac{l}{\lambda_0}$ and violated when $T > \lambda_0$.

Dirichlet and Neumann Type Boundary Value Problems for Second Order Linear Differential Equations

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On a finite interval $[a,b]$, we consider the differential equation

$$u'' = p(t)u + q(t) \tag{1}$$

with one of the following two types boundary conditions:

$$u(a) = \ell_1 u'(a) + c_1, \quad u(b) = \ell_2 u'(b) + c_2; \tag{2}$$

$$u'(a) = \ell_1 u(a) + c_1, \quad u'(b) = \ell_2 u(b) + c_2, \tag{3}$$

where $p,q : [a,b] \to \mathbb{R}$ are Lebesgue integrable functions, and ℓ_i and c_i $(i=1,2)$ are real numbers.

If $\ell_1 = \ell_2 = 0$, then problems $(1),(2)$ and $(1),(3)$ are the Dirichlet and the Neumann problems, respectively, to the investigation of which a wide literature is devoted (see, e.g. [1–3, 6, 7] and the references therein). If $|\ell_1| + |\ell_2|$ is a sufficiently small positive number, then the above mentioned problems we naturally call Dirichlet and Neumann type problems.

In case, where $\ell_1 \geq 0$, $\ell_2 \leq 0$, the optimal in a certain sense sufficient conditions for the unique solvability of problem $(1),(3)$ are established in [4].

In the general case both problem $(1),(2)$ and problem $(1),(3)$ still remain little studied. The results given in this report fill to some extent the existing gap.

Below we use the following notation.

$$[x]_+ = \frac{|x|+x}{2}\,, \quad [x]_- = \frac{|x|-x}{2}\,, \quad \|p\| = \int\limits_a^b |p(t)|\,dt.$$

Theorem 1. *If $\ell_1 \geq 0$, $\ell_2 \leq 0$, and*

$$\int\limits_a^b (t-a+\ell_1)(b-t-\ell_2)[p(t)]_-\,dt \leq b-a+\ell_1-\ell_2, \tag{4}$$

then problem $(1),(2)$ has one and only one solution.

Example 1. Introduce depending on a positive parameter γ functions

$$k_\gamma(x) = (\gamma+3)x^\gamma - x^{2\gamma+2} \ \text{ for } \ 0 \leq x \leq 1, \quad k_\gamma(x) = k_\gamma(2-x) \ \text{ for } \ 1 < x \leq 2,$$

$$v_\gamma(x) = x \exp\Big(-\frac{x^{\gamma+2}}{\gamma+2}\Big) \ \text{ for } \ 0 \leq x \leq 1, \quad v_\gamma(x) = v_\gamma(2-x) \ \text{ for } \ 1 < x \leq 2.$$

For an arbitrarily fixed $\varepsilon \in]0,1[$, set

$$\gamma = \frac{2}{\varepsilon}, \quad \ell_1 = (b-a)/(2-2^{-\gamma-1}) \ \text{ for } \ \ell_2 = -\ell_1,$$
$$p(t) = -(b-a)^{-2} k_\gamma \Big(\frac{2(t-a)+b-a}{2(b-a)} \Big) \ \text{ for } \ a \le t \le b.$$

Then

$$\int_a^b (t-a+\ell_1)(b-t-\ell_2)[p(t)]_- \, dt < (1+\varepsilon)(b-a+\ell_1-\ell_2). \tag{5}$$

On the other hand, the homogeneous problem

$$u'' = p(t)u, \tag{1_0}$$
$$u(a) = \ell_1 u'(a), \quad u(b) = \ell_2 u'(b) \tag{2_0}$$

has a nontrivial solution

$$u(t) \equiv v_\gamma \Big(\frac{2(t-a)+b-a}{2(b-a)} \Big).$$

Consequently, condition (4) is unimprovable in the sense that it cannot be replaced by condition (5) no matter how small $\varepsilon > 0$ is.

Theorem 2. *If the inequalities*

$$\ell_1 \ge 0, \ \ 0 < \ell_2 \le b-a+\ell_1, \ \ b-a+\ell_1-\ell_2+\|p\| > 0, \tag{6}$$
$$\int_a^b (t-a+\ell_1)[p(t)]_- \, dt \le 1, \quad \ell_2 \int_a^b (t-a+\ell_1)[p(t)]_+ \, dt \le b-a+\ell_1-\ell_2 \tag{7}$$

hold, then problem (1), (2) *has one and only one solution.*

Remark 1. Condition (6) cannot be replaced by the condition

$$\ell_1 \ge 0, \ \ 0 < \ell_2 \le b-a+\ell_1,$$

since if $\ell_2 = b-a+\ell_1$ and $\|p\| = 0$, then problem $(1_0), (2_0)$ has a nontrivial solution

$$u(t) \equiv \ell_1 + t - a.$$

Example 2. Let

$$\varepsilon \in]0,1[, \ \ \ell_1 = \ell_2 = \frac{b-a}{\varepsilon}, \ \ p(t) \equiv \Big(\frac{\varepsilon}{b-a} \Big)^2.$$

Then along with (6) the condition

$$\int_a^b (t-a+\ell_1)[p(t)]_- \, dt < 1, \quad \ell_2 \int_a^b (t-a+\ell_1)[p(t)]_+ \, dt < (1+\varepsilon)(b-a+\ell_1-\ell_2) \tag{8}$$

is satisfied. Nevertheless, problem $(1_0), (2_0)$ has a nontrivial solution

$$u(t) \equiv \exp \Big(\frac{\varepsilon(t-a)}{b-a} \Big).$$

Therefore, condition (7) cannot be replaced by condition (8) no matter how small $\varepsilon > 0$ is.

Theorem 3. *If the conditions*

$$\ell_2 \ge \ell_1 \ge 0, \quad \ell_2 + \|p\| > 0, \tag{9}$$

$$\int_a^b (1+\ell_1(t-a))(b-t)[p(t)]_- \, dt \le 1 + \ell_1(b-a), \tag{10}$$

$$\ell_1 + \int_a^b (1+\ell_2(t-a))[p(t)]_+ \, dt \le \ell_2 \tag{11}$$

are satisfied, then problem (1), (3) *has one and only one solution.*

Remark 2. The inequality $\ell_2 + \|p\| > 0$ cannot be omitted from condition (9). Indeed, if $\|p\| = 0$ and $\ell_2 = \ell_1 = 0$, then conditions (10), (11) hold but nevertheless equation (1_0) has a nontrivial solution

$$u(t) \equiv 1,$$

satisfying the homogeneous boundary conditions

$$u'(a) = \ell_1 u(a), \quad u'(b) = \ell_2 u(b). \tag{3_0}$$

Example 3. Introduce the function

$$r(x) = \frac{\exp(x) - \exp(-x)}{x(\exp(x) + \exp(-x))} \ \text{ for } \ x \ge 0.$$

For an arbitrarily given $\varepsilon \in]0,1[$, we choose $\delta > 0$ such that

$$1 + \delta^2 r(\delta) < (1+\varepsilon) r(\delta).$$

Let

$$p(t) \equiv \Big(\frac{\delta}{b-a}\Big)^2, \quad \ell_1 = 0, \quad \ell_2 = \frac{r(\delta)}{b-a}\,\delta^2.$$

Then conditions (9), (10) hold, and instead of (11) the inequality

$$\ell_1 + \int_a^b (1+\ell_2(t-a))[p(t)]_+ \, dt < (1+\varepsilon)\ell_2 \tag{12}$$

is satisfied. On the other hand, the homogeneous problem (1_0), (3_0) has a nontrivial solution

$$u(t) \equiv \exp\Big(\frac{\delta(t-a)}{b-a}\Big) + \exp\Big(-\frac{\delta(t-a)}{b-a}\Big).$$

Consequently, condition (10) is unimprovable in the sense that it cannot be replaced by condition (12) no matter how small $\varepsilon > 0$ is.

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On a Number of Zeros of Nontrivial Solutions to Second Order Singular Linear Differential Equations

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On a finite interval $]a,b[\,$, we consider the linear differential equation

$$u'' = p(t)u, \tag{1}$$

where $p :]a,b[\to \mathbb{R}$ is a measurable function, satisfying the condition

$$\int\limits_a^b (t-a)(b-t)|p(t)|\,dt < +\infty. \tag{2}$$

We are mainly interested in the case where the function p has nonintegrable singularity at least at one of the boundary points of the interval $]a,b[\,$, i.e. the case, where

$$\int\limits_a^b |p(t)|\,dt = +\infty.$$

A continuous function $u : [a,b] \to \mathbb{R}$ is said to be a solution to equation (1) if it is absolutely continuous together with u' on every closed interval contained in $]a,b[$ and satisfies equation (1) almost everywhere on $]a,b[\,$.

Following A. Wintner [5], we call equation (1) to be **disconjugate** on $[a,b]$ if its every nontrivial solution has no more than one zero on this interval.

In this report, we give unimprovable in a certain sense conditions under which equation (1) is disconjugate on $[a,b]$, or every its nontrivial solution has no more than two zeros on $[a,b]$. They are generalizations of the classical results by Lyapunov [4] and Hartman-Wintner [2] (see also [1], Ch. XI, § 5).

We use the following notations.

$$[x]_- = \frac{|x|-x}{2};$$

$C([a,b])$ and $L([a,b])$ are the spaces of continuous on $[a,b]$ and Lebesgue integrable on $[a,b]$ real functions, respectively;

$L_{loc}(]a,b[)$ is the space of real functions which are Lebesgue integrable on every closed interval contained in $]a,b[\,$;

$$I_1(p) = \int\limits_a^{t_1} (t-a)[p(t)]_-\,dt, \quad I_2(p) = \int\limits_a^{t_2} \frac{(t-a)(t_2-t)}{t_2-a}[p(t)]_-\,dt,$$

where the numbers $t_1 \in]a,b[$ and $t_2 \in]a,b[$ are chosen so that

$$\int\limits_a^{t_1} (t-a)[p(t)]_- \, dt = \int\limits_{t_1}^b (b-t)[p(t)]_- \, dt,$$

$$\int\limits_a^{t_2} \frac{(t-a)(t_2-t)}{t_2-a} [p(t)]_- \, dt = \int\limits_{t_2}^b \frac{(t-t_2)(b-t)}{b-t_2} [p(t)]_- \, dt. \tag{3}$$

If for some $t_0 \neq t_1$ the equality

$$\int\limits_a^{t_0} (t-a)[p(t)]_- \, dt = \int\limits_{t_0}^b (b-t)[p(t)]_- \, dt$$

is satisfied, then

$$\int\limits_{t_0}^{t_1} (t-a)[p(t)]_- \, dt = 0.$$

Consequently, for every function $p \in L_{loc}(]a,b[)$, satisfying condition (2), the number $I_1(p)$ is defined uniquely.

If $[p(t)]_- \not\equiv 0$, then the number t_2 is defined uniquely from equality (3), and thus the number $I_2(p)$ is defined uniquely as well.

Moreover, if $[p(t)]_- \not\equiv 0$ and (2) holds, then

$$I_1(p) < \int\limits_a^b \frac{(t-a)(b-t)}{b-a} [p(t)]_- \, dt. \tag{4}$$

If $[p(t)]_- \not\equiv 0$ and $p \in L([a,b])$, then

$$I_1(p) < \frac{b-a}{4} \int\limits_a^b [p(t)]_- \, dt, \quad I_2(p) < \frac{b-a}{16} \int\limits_a^b [p(t)]_- \, dt. \tag{5}$$

It has been proved by A. M. Lyapunov [4] that if $p \in C([a,b])$ and

$$\int\limits_a^b [p(t)]_- \, dt \leq \frac{4}{b-a}\,, \tag{6}$$

then equation (1) is disconjugate. Hence it easily follows that if

$$\int\limits_a^b [p(t)]_- \, dt \leq \frac{16}{b-a}\,,$$

then every nontrivial solution to equation (1) has no more than two zeros.

It has been shown by P. Hartman and A. Wintner [2] that equation (1) is disconjugate if $p \in C([a,b])$ and instead of (6) the more general condition

$$\int\limits_a^b (t-a)(b-t)[p(t)]_- \, dt \leq b-a$$

is satisfied. This result is valid also for a singular case, when the function $p \in L_{loc}(]a,b[)$ satisfies condition (2) (see [3], Lemma 2.5).

We prove the following theorems.

Theorem 1. *If along with* (2) *the condition*

$$I_1(p) \leq 1 \tag{7}$$

holds, then equation (1) *is disconjugate.*

Theorem 2. *If along with* (2) *the condition*

$$I_2(p) \leq 1 \tag{8}$$

holds, then every nontrivial solution to equation (1) *has no more than two zeros on* $[a,b]$.

According to inequalities (4) and (5), Theorems 1 and 2 are generalizations of the above mentioned results by Lyapunov and Hartman-Wintner.

Remark 1. Inequality (7) in Theorem 1 (inequality (8) in Theorem 2) is unimprovable in the sense that it cannot be replaced by the inequality $I_1(p) < 1+\varepsilon$ (by the inequality $I_2(p) < 1+\varepsilon$) no matter how small $\varepsilon > 0$ would be.

Remark 2. For inequality (7) to be satisfied, it is sufficient that for some $t_0 \in]a,b[$ the inequalities

$$\int\limits_a^{t_0} (t-a)[p(t)]_-\,dt \leq 1, \quad \int\limits_{t_0}^{b} (b-t)[p(t)]_-\,dt \leq 1$$

hold. And if for some $t_0 \in]a,b[$ the inequalities

$$\int\limits_a^{t_0} (t-a)(t_0-t)[p(t)]_-\,dt \leq t_0-a, \quad \int\limits_a^{t_0} (t-t_0)(b-t)[p(t)]_-\,dt \leq b-t_0$$

are satisfied, then inequality (8) is also satisfied.

Theorems 1 and 2 yield new and optimal in a certain sense conditions guaranteeing the unique solvability of the Dirichlet singular boundary value problem

$$u'' = p(t)u + q(t), \tag{9}$$
$$u(a) = c_1, \quad u(b) = c_2, \tag{10}$$

where p, $q \in L_{loc}(]a,b[)$, and $c_i \in \mathbb{R}$ $(i=1,2)$.

Theorem 3. *If along with* (2) *and* (7) *the condition*

$$\int\limits_a^b (t-a)(b-t)|q(t)|\,dt < +\infty \tag{11}$$

is satisfied, then problem (9), (10) *has one and only one solution.*

Example 1. Let $\alpha < 2$,

$$\delta = (2-\alpha)\Big(\frac{2}{b-a}\Big)^{2-\alpha}, \quad p(t) = -\delta\bigg(\frac{b-a-|b+a-2t|}{2}\bigg)^{-\alpha} \ \text{ for } \ a < t < b,$$

and let q be the function satisfying condition (11). Then

$$\int\limits_a^b (t-a)(b-t)[p(t)]_- \, dt = \frac{4-\alpha}{3-\alpha}\,(b-a) > b-a,$$

i.e. the Lyapunov–Hartman–Wintner condition is violated. On the other hand,

$$I_1(p) = 1,$$

and by Theorem 3 problem (9), (10) is uniquely solvable.

Theorem 4. *Let conditions* (2), (8), *and* (11) *hold and there exist a function* $p_0 \in L_{loc}(]a,b[)$ *such that*

$$p(t) \le p_0(t) \le 0 \ \textit{ for } \ a < t < b, \quad \operatorname{mes}\big\{t \in]a,b[\, : \ p(t) < p_0(t)\big\} > 0,$$

and the boundary value problem

$$u'' = p_0(t)u; \ \ u(a) = 0, \ \ u(b) = 0$$

has a positive on the open interval $]a,b[$ *solution. Then problem* (9), (10) *has one and only one solution.*

Corollary 1. *Let*

$$p(t) < -\Big(\frac{\pi}{b-a}\Big)^2 \ \textit{ for } \ a < t < b,$$

and let conditions (2), (8), *and* (11) *be satisfied. Then problem* (9), (10) *has one and only one solution.*

Example 2. Let $\alpha < -2$,

$$0 < \delta \le \Big(1 - \frac{\pi^2}{24}\Big)(2-\alpha)\Big(\frac{2}{b-a}\Big)^{2-\alpha},$$

$$p(t) = -\Big(\frac{\pi}{b-a}\Big)^2 - \delta\bigg(\frac{b-a-|b+a-2t|}{2}\bigg)^{-\alpha} \ \text{ for } \ a < t < b,$$

and let q be the function satisfying condition (11). Then $I_2(p) \le 1$, and according to the above corollary problem (9), (10) has one and only one solution.

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On a Dirichlet Type Boundary Value Problem for a Class of Linear Partial Differential Equations

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Let $\Omega = (0, \omega_1) \times (0, \omega_2) \times (0, \omega_3)$ be an open rectangular box, and let E be an *orthogonally convex* piecewise smooth domain inscribed in Ω.

A set $G \in \mathbb{R}^n$ is defined to be ***orthogonally convex*** if, for every line L that is parallel to one of standard basis vectors, the intersection of G with L is empty, a point, or a single segment.

In the domain E consider the boundary value problem

$$u^{(2,2,2)} = \sum_{\alpha < \mathbf{2}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{1}$$

$$u\,\nu_1\big|_{\partial E} = \nu_1(\mathbf{x})\psi_1(\mathbf{x}), \quad u^{(2,0,0)}\,\nu_2\big|_{\partial E} = \nu_2(\mathbf{x})\psi_2(\mathbf{x}), \quad u^{(2,2,0)}\,\nu_3\big|_{\partial E} = \nu_3(\mathbf{x})\psi_3(\mathbf{x}). \tag{2}$$

Here $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{2} = (2, 2, 2)$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ is a multi-index,

$$u^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{\alpha_1+\alpha_2+\alpha_3} u(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}\,,$$

∂E is the boundary of E, and $\boldsymbol{\nu}(\mathbf{x}) = (\nu_1(\mathbf{x}), \nu_2(\mathbf{x}), \nu_2(\mathbf{x}))$ is the outward unit normal vector at point $\mathbf{x} \in \partial E$, $p_{\boldsymbol{\alpha}} \in C(\overline{E})$ $(\boldsymbol{\alpha} < \mathbf{2})$, $q \in C(\overline{E})$, $\psi_i \in C^{2,2,2}(\overline{E})$ and $\overline{E}$ is the closure of E.

By a solution of problem (1),(2) we understand a *classical* solution, i.e., a function $u \in C^{2,2,2}(E) \cap C^{2,2,0}(\overline{E})$ satisfying equation (1) and the boundary conditions (2) everywhere in E and ∂E, respectively.

$C^{2,2,2}(E)$ is the space of continuous functions $u : E \to \mathbb{R}$ having continuous partial derivatives $u^{(\boldsymbol{\alpha})}$ $(\boldsymbol{\alpha} \leq \mathbf{2})$.

Throughout the paper the following notations will be used.

$\mathbf{0} = (0, 0, 0)$, $\mathbf{1} = (1, 1, 1)$.

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) < \boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) \iff \alpha_i \leq \beta_i$ $(i = 1, 2, 3)$ and $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$.

$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \leq \boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3) \iff \boldsymbol{\alpha} < \boldsymbol{\beta}$, or $\boldsymbol{\alpha} = \boldsymbol{\beta}$.

$\|\boldsymbol{\alpha}\| = |\alpha_1| + |\alpha_2| + |\alpha_3|$.

$\boldsymbol{\Upsilon}_{\mathbf{2}} = \big\{\boldsymbol{\alpha} < \mathbf{2} :\ \alpha_i = 2 \text{ for some } i \in \{1, 2, 3\}\big\}$.

$\boldsymbol{O}_{\mathbf{2}} = \big\{\boldsymbol{\alpha} < \mathbf{2} :\ \|\boldsymbol{\alpha}\| \text{ is odd}\big\}$.

$\operatorname{supp} \boldsymbol{\alpha} = \{i \mid\ \alpha_i > 0\}$.

$\mathbf{x}_{\boldsymbol{\alpha}} = (\chi(\alpha_1)x_1, \chi(\alpha_2)x_2, \chi(\alpha_3)x_3)$, where $\chi(\alpha) = 0$ if $\alpha = 0$, and $\chi(\alpha) = 1$ if $\alpha > 0$.

$\mathbf{x}_{\boldsymbol{\alpha}}$ will be identified with $(x_{i_1}, \dots, x_{i_l})$, where $\{i_1, \dots, i_l\} = \operatorname{supp} \boldsymbol{\alpha}$.

$\widehat{\mathbf{x}}_{\boldsymbol{\alpha}} = \mathbf{x} - \mathbf{x}_{\boldsymbol{\alpha}}$.

$$f^+(z) = \frac{f(z) + |f(z)|}{2}\,, \quad f^-(z) = \frac{|f(z)| - f(z)}{2}\,.$$

$H(f)(\mathbf{x})$ is the Hessian matrix of function f at point $\mathbf{x}$.

Along with problem (1), (2) consider the corresponding homogeneous problem

$$u^{(2,2,2)} = \sum_{\alpha<\mathbf{2}} p_{\boldsymbol{\alpha}}(\mathbf{x}) u^{(\boldsymbol{\alpha})}, \tag{1_0}$$

$$u\,\nu_1\big|_{\partial E} = 0, \quad u^{(2,0,0)}\,\nu_2\big|_{\partial E} = 0, \quad u^{(2,2,0)}\,\nu_3\big|_{\partial E} = 0. \tag{2_0}$$

Two-dimensional versions of problem (1), (2) were studied in [1–4]. The case of a characteristic rectangle was considered [1] and [2]. In [3] and [4] two-dimensional problems were considered in a orthogonally convex smooth domains.

Orthogonal convexity and smoothness of a domain are essential requirements and cannot be relaxed. Examples attesting the paramount importance of orthogonal convexity and smoothness of a domain were introduced in Remarks 1 and 2 of [4]. Similar examples can be easily constructed for the three-dimensional case.

Characteristic rectangles were the only admissible piecewise smooth domains for two-dimensional problems. In the three-dimensional case admissible piecewise smooth domains consist of characteristic rectangular boxes and right cylinders with an orthogonally convex smooth base.

We study problem (1), (2) in the following three cases: characteristic rectangular box; a right cylinder with an orthogonally convex smooth base; an *orthogonally convex* smooth domain.

It is not difficult to show that the problem

$$u^{(2,2,2)} = 0,$$

$$u\,\nu_1\big|_{\partial E} = \nu_1(\mathbf{x})\psi_1(\mathbf{x}), \quad u^{(2,0,0)}\,\nu_2\big|_{\partial E} = \nu_2(\mathbf{x})\psi_2(\mathbf{x}), \quad u^{(2,2,0)}\,\nu_3\big|_{\partial E} = \nu_3(\mathbf{x})\psi_3(\mathbf{x})$$

is uniquely solvable in all three aforementioned cases. Consequently, without loss of generality, problem (1), (2) can always be reduced to the problem with the zero boundary conditions.

Due to this fact, for the sake of technical simplicity, all results will be formulated for problem (1), (2_0).

Case I: Characteristic Rectangular Box. Let $E = \Omega$. For the rectangular box Ω the boundary conditions (2_0) receive the form

$$u(\sigma\,\omega_1, x_1, x_2) = 0, \quad u^{(2,0,0)}(x_1, \sigma\,\omega_2, x_2) = 0, \quad u^{(2,2,0)}(x_1, x_2, \sigma\,\omega_3) = 0 \ \ (\sigma = 0, 1).$$

It is easy to see that the latter conditions are equivalent to the following ones

$$u(\sigma\,\omega_1, x_1, x_2) = 0, \quad u(x_1, \sigma\,\omega_2, x_2) = 0, \quad u(x_1, x_2, \sigma\,\omega_3) = 0 \ \ (\sigma = 0, 1). \tag{3}$$

Theorem 1. *Let*

$$p_{\boldsymbol{\alpha}}(\mathbf{x}) \equiv p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}) \ \ \textit{if} \ \ \boldsymbol{\alpha} \in \boldsymbol{\Upsilon}_{\mathbf{2}} \cap \boldsymbol{O}_{\mathbf{2}},$$

and let the following inequalities hold:

$$p_{220}(\mathbf{x}) \equiv p_{220}(x_3) > -\frac{\pi^2}{\omega_3^2}, \quad p_{202}(\mathbf{x}) \equiv p_{202}(x_2) > -\frac{\pi^2}{\omega_2^2}, \quad p_{022}(\mathbf{x}) \equiv p_{022}(x_1) > -\frac{\pi^2}{\omega_1^2}, \tag{4}$$

$$p_{220}^{-}(x_3)\,\frac{\omega_3^2}{\pi^2} + p_{202}^{-}(x_2)\,\frac{\omega_2^2}{\pi^2} + p_{200}^{+}(\mathbf{x})\,\frac{\omega_2^2\omega_3^2}{\pi^4} + \big|p_{211}(\mathbf{x})\big|\,\frac{\omega_2\omega_3}{\pi^2} < 1, \tag{5}$$

$$p_{220}^{-}(x_3)\,\frac{\omega_3^2}{\pi^2} + p_{022}^{-}(x_1)\,\frac{\omega_1^2}{\pi^2} + p_{020}^{+}(\mathbf{x})\,\frac{\omega_1^2\omega_3^2}{\pi^4} + \big|p_{121}(\mathbf{x})\big|\,\frac{\omega_1\omega_3}{\pi^2} < 1, \tag{6}$$

$$p_{202}^{-}(x_2)\,\frac{\omega_2^2}{\pi^2} + p_{022}^{-}(x_1)\,\frac{\omega_1^2}{\pi^2} + p_{002}^{+}(\mathbf{x})\,\frac{\omega_1^2\omega_2^2}{\pi^4} + \big|p_{112}(\mathbf{x})\big|\,\frac{\omega_1\omega_2}{\pi^2} < 1. \tag{7}$$

Then problem (1), (3) *has the Fredholm property, i.e.:*

(i) *problem* $(1_0),(3)$ *has a finite dimensional space of solutions;*

(ii) *problem* $(1),(3)$ *is uniquely solvable if and only if problem* $(1_0),(3)$ *has only the trivial solution.*

Furthermore, every solution of problem $(1),(3)$ *belongs to* $C^{2,2,2}(\overline{\Omega})$.

Remark 1. The strict inequalities (4)–(7) are sharp and cannot be replaced by nonstrict ones. Violation of at least one of the above inequalities can lead to the loss of the Fredholm property of problem $(1),(2)$. To verify this, consider the problem

$$u^{(2,2,2)}=(-1)^{\|\boldsymbol{\alpha}\|}u^{(2\boldsymbol{\alpha})}+u-\sin x_1\sin x_2\sin x_3\,q(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}), \tag{8}$$

$$u(\sigma\,\pi,x_1,x_2)=0,\quad u(x_1,\sigma\,\pi,x_2)=0,\quad u(x_1,x_2,\sigma\,\pi)=0\ \ (\sigma=0,1) \tag{9}$$

in the domain $E=(0,\pi)\times(0,\pi)\times(0,\pi)$. Here $\mathbf{0}<\boldsymbol{\alpha}<\mathbf{1}$, and q is an arbitrary ***non-differentiable*** continuous function. The problem satisfies all of the inequalities (4)–(7) except the one for the coefficient $p_{2\boldsymbol{\alpha}}$: instead of $(-1)^{\|\boldsymbol{\alpha}\|}p_{2\boldsymbol{\alpha}}>1$ we have $(-1)^{\|\boldsymbol{\alpha}\|}p_{2\boldsymbol{\alpha}}=1$. As a result, problem $(8),(9)$ does not have the Fredholm property. Indeed, despite the fact that the homogeneous problem

$$u^{(2,2,2)}=(-1)^{\|\boldsymbol{\alpha}\|}u^{(2\boldsymbol{\alpha})}+u,$$
$$u(\sigma\,\pi,x_1,x_2)=0,\quad u(x_1,\sigma\,\pi,x_2)=0,\quad u(x_1,x_2,\sigma\,\pi)=0\ \ (\sigma=0,1)$$

has only the trivial solution, problem $(8),(9)$ has the unique weak solution

$$u(x)=\sin x_1\sin x_2\sin x_3\,q(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}}),$$

which is not a classical solution due to ***non-differentiability*** of the function q.

Consider the equation

$$u^{(2,2,2)}=\sum_{\boldsymbol{\alpha}<\mathbf{1}}p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})u^{(2\boldsymbol{\alpha})}+\sum_{\alpha\in\boldsymbol{O_2}}p_{\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}). \tag{10}$$

Corollary 1. *Let*

$$(-1)^{\|\boldsymbol{\alpha}\|}p_{2\boldsymbol{\alpha}}(\widehat{\mathbf{x}}_{\boldsymbol{\alpha}})\geq 0\ \ \textit{for}\ \ \boldsymbol{\alpha}<\mathbf{1}. \tag{11}$$

Then problem $(10),(3)$ *is uniquely solvable.*

Case II: Right Cylinder. Let $E=\{(x_1,x_2,x_3)\in\Omega:\ (x_1,x_2)\in G,\ x_3\in(0,\omega_3)\}$, where G is an *orthogonally convex* open domain with C^2 boundary inscribed in the rectangle $(0,\omega_1)\times(0,\omega_2)$, i.e.,

$$\begin{aligned}G=\big\{(x_1,x_2)\in\Omega:\ x_1\in(0,\omega_1),\ x_2\in(\gamma_1(x_1),\gamma_2(x_1))\big\}\\ =\big\{(x_1,x_2)\in\Omega:\ x_2\in(0,\omega_2),\ x_1\in(\eta_1(x_2),\eta_2(x_2))\big\},\end{aligned} \tag{12}$$

and $\gamma_i\in C([0,\omega_1])\cap C^2((0,\omega_1))$, $\eta_i\in C([0,\omega_2])\cap C^2((0,\omega_2))$ $(i=1,2)$.

In the right cylinder E consider the following equations

$$\begin{aligned}u^{(2,2,2)}&=p_{220}(x_3)u^{(2,2,0)}+p_{202}(x_2)u^{(2,0,2)}+p_{200}(\mathbf{x})u^{(2,0,0)}+p_{020}(\mathbf{x})u^{(0,2,0)}+p_{002}(x_2,x_3)u^{(0,0,2)}\\&\quad+\sum_{\alpha\leq\mathbf{1}}p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})}+q(\mathbf{x}),\end{aligned} \tag{13}$$

$$\begin{aligned}u^{(2,2,2)}&=p_{220}(x_3)u^{(2,2,0)}+p_{202}(x_2)u^{(2,0,2)}+p_{022}(x_1)u^{(0,2,2)}\\&\quad+p_{200}(\mathbf{x})u^{(2,0,0)}+p_{020}(x_3)u^{(0,2,0)}+p_{002}(x_2)u^{(0,0,2)}+p_{000}(x_2,x_3)u+q(\mathbf{x})\end{aligned} \tag{14}$$

and

$$\begin{aligned}u^{(2,2,2)} &= p_{220}(x_3)u^{(2,2,0)} + p_{202}(x_2)u^{(2,0,2)} + p_{022}(x_1)u^{(0,2,2)}\\ &\quad + p_{200}(\mathbf{x})u^{(2,2,0)} + p_{020}(x_3)u^{(0,2,0)} + p_{002}(x_2)u^{(0,0,2)}\\ &\quad + \sum_{\alpha\le\mathbf{1}} p_{\boldsymbol\alpha}(\mathbf{x})u^{(\boldsymbol\alpha)} + q(\mathbf{x}).\end{aligned}\tag{15}$$

In view of (12), conditions (2_0) receive the form

$$\begin{aligned}u(\zeta_i(x_2,x_3),x_2,x_3) = 0, \quad u^{(2,0,0)}(x_1,\eta_i(x_1,x_3),x_3) = 0,&\\ u^{(2,2,0)}(x_1,x_2,\gamma_i(x_1,x_2)) = 0 \ \ (i=1,2).&\end{aligned}\tag{16}$$

Theorem 2. *Let the following inequalities hold:*

$$p_{220}(x_3)\ge 0, \quad p_{202}(x_2)\ge 0, \quad p_{200}(\mathbf{x})\le 0, \quad p_{020}(\mathbf{x})\le 0, \quad p_{002}(x_2,x_3)\le 0.$$

Then problem (13), (16) *has the Fredholm property.*

Theorem 3. *Let G be a **convex** domain, i.e.,*

$$(-1)^{i-1}\gamma_i''(x_1)\ge 0 \ \ \textit{for} \ \ x_1\in(0,\omega_1) \ \ (i=1,2)\tag{17}$$

and

$$(-1)^{i-1}\eta_i''(x_2)\ge 0 \ \ \textit{for} \ \ x_2\in(0,\omega_2) \ \ (i=1,2),\tag{18}$$

and let

$$p_{220}(x_3)\ge 0, \quad p_{202}(x_2)\ge 0, \quad p_{202}(x_1)\ge 0,\tag{19}$$

$$\begin{gathered}p_{200}(\mathbf{x})\le 0, \quad p_{020}(x_3)\le 0, \quad p_{002}(x_2)\le 0,\\ p_{000}(x_2,x_3)\ge 0.\end{gathered}\tag{20}$$

Then problem (14), (16) *is uniquely solvable.*
*Furthermore, if G is **strongly convex**, i.e.,*

$$(-1)^{i-1}\gamma_i''(x_1) > 0 \ \ \textit{for} \ \ x_1\in(0,\omega_1) \ \ (i=1,2)\tag{21}$$

and

$$(-1)^{i-1}\eta_i''(x_2) > 0 \ \ \textit{for} \ \ x_2\in(0,\omega_2) \ \ (i=1,2),\tag{22}$$

then the solution of problem (14), (16) *belongs to* $C^{2,2,2}(\overline{E})$.

Corollary 2. *Let inequalities* (17)–(20) *hold. Then problem* (15), (16) *has the Fredholm property. Furthermore, if inequalities* (21) *and* (22) *hold, then every solution of problem* (15), (16) *belongs to* $C^{2,2,2}(\overline{E})$.

Case III: Smooth Domain. Let E be an *orthogonally convex* open domain with $C^{2,2}$ boundary inscribed in the characteristic box Ω, i.e.,

$$\begin{aligned}E &= \big\{(x_1,x_2,x_3)\in\Omega: \ (x_1,x_2)\in G_{12}, \ x_3\in(\gamma_1(x_1,x_2),\gamma_2(x_1,x_2))\big\}\\ &= \big\{(x_1,x_2,x_3)\in\Omega: \ (x_1,x_3)\in G_{13}, \ x_2\in(\eta_1(x_1,x_3),\eta_2(x_1,x_3))\big\}\\ &= \big\{(x_1,x_2,x_3)\in\Omega: \ (x_2,x_3)\in G_{13}, \ x_1\in(\zeta_1(x_2,x_3),\zeta_2(x_2,x_3))\big\},\end{aligned}\tag{23}$$

where $\gamma_i \in C(\overline{G}_{12}) \cap C^{2,2}(G_{12})$, $\eta_i \in C(\overline{G}_{13}) \cap C^{2,2}(G_{13})$, $\zeta_i \in C(\overline{G}_{23}) \cap C^{2,2}(G_{23})$ $(i=1,2)$, and G_{12}, G_{13} and G_{23} are orthogonally convex smooth open domains inscribed in $(0,\omega_1)\times(0,\omega_2)$, $(0,\omega_1)\times(0,\omega_3)$ and $(0,\omega_2)\times(0,\omega_3)$, respectively.

In the domain E consider the following equations:

$$u^{(2,2,2)} = p_{220}(\mathbf{x})u^{(2,2,0)} + p_{200}(x_1,x_3)u^{(2,0,0)} + \sum_{\alpha\leq\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}), \tag{24}$$

$$\begin{aligned} u^{(2,2,2)} &= p_{220}(x_3)u^{(2,2,0)} + p_{202}(x_2)u^{(2,0,2)} \\ &\quad + p_{200}(\mathbf{x})u^{(2,0,0)} + p_{020}(x_3)u^{(0,2,0)} + p_{002}(x_2)u^{(0,0,2)} + p_{000}(x_2,x_3)u + q(\mathbf{x}) \end{aligned} \tag{25}$$

and

$$\begin{aligned} u^{(2,2,2)} &= p_{220}(x_3)u^{(2,2,0)} + p_{202}(x_2)u^{(2,0,2)} + p_{200}(\mathbf{x})u^{(2,2,0)} + p_{020}(x_3)u^{(0,2,0)} + p_{002}(x_2)u^{(0,0,2)} \\ &\quad + \sum_{\alpha\leq\mathbf{1}} p_{\boldsymbol{\alpha}}(\mathbf{x})u^{(\boldsymbol{\alpha})} + q(\mathbf{x}). \end{aligned} \tag{26}$$

In view of (23), conditions (2_0) receive the form

$$\begin{aligned} &u(\zeta_i(x_2,x_3),x_2,x_3)=0, \quad u^{(2,0,0)}(x_1,\eta_i(x_1,x_3),x_3)=0, \\ &\qquad\qquad u^{(2,2,0)}(x_1,x_2,\gamma_i(x_1,x_2))=0 \ \ (i=1,2). \end{aligned} \tag{27}$$

Theorem 4. *Let the following inequalities hold:*

$$\begin{gathered} p_{220}(\mathbf{x}) \geq 0, \\ p_{200}(x_1,x_3) \leq 0. \end{gathered}$$

Then problem (24), (27) *has the Fredholm property.*

Theorem 5. *Let E be a* ***convex*** *domain, i.e., let*

$$(-1)^{i-1}H[\gamma_i](x_1,x_2) \ \text{ be positive semi-definite for } (x_1,x_2)\in G_{12} \ \ (i=1,2), \tag{28}$$

$$(-1)^{i-1}H[\eta_i](x_1,x_3) \ \text{ be positive semi-definite for } (x_1,x_3)\in G_{13} \ \ (i=1,2), \tag{29}$$

$$(-1)^{i-1}H[\zeta_i](x_2,x_3) \ \text{ be positive semi-definite for } (x_2,x_3)\in G_{23} \ \ (i=1,2), \tag{30}$$

and let

$$p_{220}(x_3)\geq 0, \quad p_{202}(x_2)\geq 0, \tag{31}$$

$$\begin{gathered} p_{200}(\mathbf{x})\leq 0, \quad p_{020}(x_3)\leq 0, \quad p_{002}(x_2)\leq 0, \\ p_{000}(x_2,x_3)\geq 0. \end{gathered} \tag{32}$$

Then problem (25), (27) *is uniquely solvable.*
Furthermore, if E is ***strongly convex****, i.e.,*

$$(-1)^{i-1}H[\gamma_i](x_1,x_2) \ \text{ is positive definite for } (x_1,x_2)\in G_{12} \ \ (i=1,2), \tag{33}$$

$$(-1)^{i-1}H[\eta_i](x_1,x_3) \ \text{ is positive definite for } (x_1,x_3)\in G_{13} \ \ (i=1,2), \tag{34}$$

$$(-1)^{i-1}H[\zeta_i](x_2,x_3) \ \text{ is positive definite for } (x_2,x_3)\in G_{23} \ \ (i=1,2), \tag{35}$$

then the solution of problem (25), (27) *belongs to* $C^{2,2,2}(\overline{E})$.

Corollary 3. *Let conditions* (28)–(32) *hold. Then problem* (26), (27) *has the Fredholm property. Furthermore, if conditions* (33)–(35) *hold, then every solution of problem* (26), (27) *belongs to* $C^{2,2,2}(\overline{E})$.

Remark 2. In a strongly convex domain the boundary conditions (2) are equivalent to the boundary conditions

$$u\big|_{\partial E} = \psi_1(\mathbf{x}), \quad u^{(2,0,0)}\big|_{\partial E} = \psi_2(\mathbf{x}), \quad u^{(2,2,0)}\big|_{\partial E} = \psi_3(\mathbf{x}).$$

Remark 3. Without the requirement that the domain E be strongly convex the solution of problem (1), (2) may not belong to $C^{2,2,2}(\overline{E})$.

As an example, in the domain $E = \{(x_1, x_2, x_3) : \ x_1^4 + x_2^4 + x_3^4 < 1\}$ consider the problem

$$u^{(2,2,2)} = 0, \tag{36}$$

$$u\big|_{\partial E} = 0, \quad u^{(2,0,0)}\big|_{\partial E} = 2, \quad u^{(2,2,0)}\big|_{\partial E} = 0. \tag{37}$$

E is a convex domain. However, E is not strongly convex, since the Hessian matrices mentioned in Theorem 5 are positive semi-definite rather than positive definite along the three "main meridians"

$$\begin{cases} x_1^4 + x_2^4 = 1 \\ x_3 = 0 \end{cases}, \quad \begin{cases} x_1^4 + x_3^4 = 1 \\ x_2 = 0 \end{cases}, \quad \text{and} \quad \begin{cases} x_2^4 + x_3^4 = 1 \\ x_1 = 0 \end{cases}.$$

As a result, the unique solution of problem (36), (37) $u(\mathbf{x}) = x_1^2 - \sqrt{1 - x_2^4 - x_3^4}$ does not belong to $C^{2,2,2}(\overline{E})$ since $u^{(0,1,0)}$ and $u^{(0,0,1)}$ are discontinuous along the third "main meridian".

It is worth noticing that problem (36), (37) considered in the unit ball $E = \{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < 1\}$ has a unique solution $u(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 - 1$ which belongs to $C^{2,2,2}(\overline{E})$. Such contrast is explained by the fact that the unit ball is a strongly convex domain.

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On One-Dimensional Nonlinear Integro-Differential System with Source Terms

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The process of electromagnetic field propagation into a substance, its mathematical modeling, investigation, and numerical solution belong to one of the most important tasks in applied mathematics. As a rule, this process is accompanied by the release of thermal energy, which causes changes in the permeability of the medium and affects the diffusion process since the coefficient of conductivity of the medium significantly depends on temperature. Mathematical simulation of the mentioned process, like many other applied problems, results in nonlinear partial differential and integro-differential equations and systems of those equations. In a quasistationary case the corresponding system of the Maxwell equations has the following form [12]:

$$\frac{\partial H}{\partial t} = -\operatorname{rot}(\nu_m \operatorname{rot} H), \tag{1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m (\operatorname{rot} H)^2, \tag{2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m characterize the heat capacity and electrical conductivity of the medium. Equation (1) describes the propagation of the magnetic field in the medium whereas equation (2) expresses a change of the temperature due to the Joule heating. Assume that coefficients of thermal heat capacity and electrical conductivity of the substance depending on temperature. In this case, as it is shown in [3], system (1), (2) can be reduced to the following nonlinear parabolic type integro-differential model

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\left[a\Big(\int_0^t |\operatorname{rot} H|^2\, d\tau\Big) \operatorname{rot} H\right]. \tag{3}$$

Let us note that the above-mentioned integro-differential model (3) is complex and only particular classes are investigated (see, for example, [1–11, 13–15, 17, 18] and the references therein). Consider the case when all three components of the magnetic field vector are functions of time and one spatial variable $H_i = H_i(x,t)$, $i = 1, 2, 3$. Thus, in this case we have:

$$\operatorname{rot} H = \Big(0, -\frac{\partial H_3}{\partial x}, \frac{\partial H_2}{\partial x}\Big),$$

$$\operatorname{rot}(a(S) \operatorname{rot} H) = \bigg(0, -\frac{\partial}{\partial x}\Big(a(S)\,\frac{\partial H_2}{\partial x}\Big), -\frac{\partial}{\partial x}\Big(a(S)\,\frac{\partial H_3}{\partial x}\Big)\bigg),$$

where

$$S(x,t) = \int_0^t \bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2 + \Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\, d\tau,$$

and system (3) takes the following form:

$$
\begin{gathered}
\frac{\partial H_1}{\partial t}=0,\\
\frac{\partial H_2}{\partial t}-\frac{\partial}{\partial x}\bigg[a\bigg(\int\limits_0^t\bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2+\Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\,d\tau\bigg)\,\frac{\partial H_2}{\partial x}\bigg]=0,\\
\frac{\partial H_3}{\partial t}-\frac{\partial}{\partial x}\bigg[a\bigg(\int\limits_0^t\bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2+\Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\,d\tau\bigg)\,\frac{\partial H_3}{\partial x}\bigg]=0.
\end{gathered}
\tag{4}
$$

Our goal is to study the convergence of the finite difference scheme for the following initial-boundary value problem posed for the nonlinear integro-differential system (4) with source terms and known right-hand sides:

$$
\begin{gathered}
\frac{\partial H_1}{\partial t}+g_1(H_1)=f_1,\\
\frac{\partial H_2}{\partial t}-\frac{\partial}{\partial x}\bigg[a\bigg(\int\limits_0^t\bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2+\Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\,d\tau\bigg)\,\frac{\partial H_2}{\partial x}\bigg]+g_2(H_2)=f_2,\\
\frac{\partial H_3}{\partial t}-\frac{\partial}{\partial x}\bigg[a\bigg(\int\limits_0^t\bigg[\Big(\frac{\partial H_2}{\partial x}\Big)^2+\Big(\frac{\partial H_3}{\partial x}\Big)^2\bigg]\,d\tau\bigg)\,\frac{\partial H_3}{\partial x}\bigg]+g_3(H_3)=f_3,
\end{gathered}
\tag{5}
$$

$$
H_2(0,t)=H_2(1,t)=H_3(0,t)=H_3(1,t)=0,\ \ t\geq 0,
\tag{6}
$$

$$
H_1(x,0)=H_{10}(x),\quad H_2(x,0)=H_{20}(x),\quad H_3(x,0)=H_{30}(x),\ \ x\in[0,1],
\tag{7}
$$

where H_{i0}, g_i, f_i, $i=1,2,3$ are given functions and g_i are monotonically increased and positively defined functions.

Due to the fact that the last two equations of system (5) are strongly connected to each other, we will consider these equation jointly, whereas the first equation will be considered independently.

Let us correspond the finite difference scheme for problem (5)–(7). On $[0,1]\times[0,T]$ let us introduce a net with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,\dots,M$; $j=0,1,\dots,N$ with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is designed by (u_i^j,v_i^j,w_i^j) and the exact solution to problem (5)–(7)) by $(H_{1i}^j,H_{2i}^j,H_{3i}^j)$. We will use the following known notations [16] of forward and backward derivatives:

$$
r_{x,i}^j=\frac{r_{i+1}^j-r_i^j}{h}\,,\quad r_{\overline{x},i}^j=\frac{r_i^j-r_{i-1}^j}{h}\,,\quad r_{t,i}^j=\frac{r_i^{j+1}-r_i^j}{\tau}\,,
$$

and inner products and corresponding norms:

$$
\begin{gathered}
(r^j,y^j)=h\sum_{i=1}^{M-1}r_i^jy_i^j,\quad (r^j,y^j]=h\sum_{i=1}^{M}r_i^jy_i^j,\\
\|r^j\|=(r^j,r^j)^{1/2},\quad \|r^j]|=(r^j,r^j]^{1/2}.
\end{gathered}
$$

For problem (5)–(7) let us consider the following finite difference scheme:

$$
\begin{gathered}
\frac{u_i^{j+1}-u_i^j}{\tau}+g_1(u_i^{j+1})=f_{1,i}^j,\\
\frac{v_i^{j+1}-v_i^j}{\tau}-\bigg\{a\bigg(\tau\sum_{k=1}^{j+1}\big[(v_{\overline{x},i}^k)^2+(w_{\overline{x},i}^k)^2\big]\bigg)v_{\overline{x},i}^{j+1}\bigg\}_{x,i}+g_2\left(v_i^{j+1}\right)=f_{2,i}^j,\\
\frac{w_i^{j+1}-w_i^j}{\tau}-\bigg\{a\bigg(\tau\sum_{k=1}^{j+1}\big[(v_{\overline{x},i}^k)^2+(w_{\overline{x},i}^k)^2\big]\bigg)w_{\overline{x},i}^{j+1}\bigg\}_{x,i}+g_3\left(w_i^{j+1}\right)=f_{3,i}^j,\\
i=1,2,\dots,M-1;\ \ j=0,1,\dots,N-1,\\
v_0^j=v_M^j=w_0^j=w_M^j=0,\ \ j=0,1,\dots,N,\\
u_i^0=H_{10,i},\quad v_i^0=H_{20,i},\quad v_i^0=H_{30,i},\ \ i=0,1,\dots,M.
\end{gathered}
\tag{8}
$$

Multiplying equations in (8) scalarly by u^{j+1}, v^{j+1} and w^{j+1}, respectively, it is not difficult to get the inequalities:

$$
\|u^n\|<C,\quad \|v^n\|^2+\sum_{j=1}^n\|v_{\overline{x}}^j]|^2\tau<C,\quad \|w^n\|^2+\sum_{j=1}^n\|w_{\overline{x}}^j]|^2\tau<C,\ \ n=1,2,\dots,N,
\tag{9}
$$

where here and below C is a positive constant independent from τ and h.

The a priori estimates (9) guarantee the stability of scheme (8). The main statement of this note can be stated as follows.

Theorem. *If $a=a(S)\geq a_0=Const>0$, $a'(S)\geq 0$, $a''(S)\leq 0$ and g_i, $i=1,2,3$ are positively defined and monotonically increased functions, and problem* (5)–(7) *has a sufficiently smooth solution, then the solution of the difference scheme* (8) *tends to the solution of the continuous problem* (5)–(7) *as $\tau\to 0$, $h\to 0$ and the following estimates are true:*

$$
\|u^j-H_1^j\|\leq C(\tau),\quad \|v^j-H_2^j\|\leq C(\tau+h),\quad \|w^j-H_3^j\|\leq C(\tau+h).
$$

We have carried out numerous numerical experiments for problem (5)–(7) with different kind of right hand sides and initial-boundary conditions. Results of numerical experiments confirmed findings in the above-stated theorem.

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On Some Positive Solutions to Differential Equations with General Power-Law Nonlinearities

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1 Introduction

Consider solutions with positive initial data to differential equation with general power-law nonlinearity

$$y^{(n)}=p(x,y,y',\dots,y^{(n-1)})|y|^{k_0}|y'|^{k_1}\cdots|y^{(n-1)}|^{k_{n-1}}\operatorname{sgn}(y\,y'\,\cdots\,y^{(n-1)}), \tag{1.1}$$

with $n\geq 2$, positive real nonlinearity exponents $k_0,k_1,\dots,k_{n-1}$ and positive continuous in x and Lipschitz continuous in $u_0,u_1,\dots,u_{n-1}$ bounded function $p(u_0,u_1,\dots,u_{n-1})$.

The results on qualitative behavior and asymptotic estimates of positive increasing solutions for higher order nonlinear differential equations were obtained by I. T. Kiguradze and T. A. Chanturia in [9]. Questions on qualitative and asymptotic behavior of solutions to higher order Emden–Fowler differential equations ($k_1=\cdots=k_{n-1}=0$) were studied by I. V. Astashova in [1, 4–6].

In the case $n=2$ the results on qualitative behavior of solutions can be found in [10], and asymptotic behavior was studied in [11].

Equation (1.1) in the case $n=3$, $k_0>0$, $k_0\neq 1$, $k_1=k_2=0$, was studied by I. Astashova in [1, Chapters 6–8]. In particular, asymptotic classification of solutions to such equations was given in [3, 6], and proved in [2]. Qualitative properties of solutions in the case $n=3$, $k_0>0$, $k_1>0$, $k_2>0$ were studied in [12]. In this paper several results are generalized for higher order differential equations with general power-law nonlinearity.

For higher order differential equations, nonlinear with respect to derivatives of solutions, the asymptotic behavior of certain types of solutions was studied by V. M. Evtukhov, A. M. Klopot in [7, 8].

2 On the behavior of solutions

For equations of second- and third-order the following results on qualitative behavior of solutions with positive initial data were obtained.

Theorem 2.1 ([10]). *Suppose $n=2$ and $k_0+k_1>1$. Let the function $p(x,u,v)$ be continuous in x, Lipschitz continuous in u, v and satisfy the inequality $p(x,u,v)\geq m>0$. Then there exists a constant $\zeta=\zeta(m,k_0,k_1)$ such that any maximally extended solution $y(x)$ to equation (1.1), satisfying at some point x_0 the conditions $y(x_0)\geq 0$, $y'(x_0)=y_1>0$, has a finite right domain boundary x^* satisfying the estimate*

$$x^*-x_0<\zeta\,y_1^{-\frac{k_0+k_1-1}{k_0+1}}.$$

Theorem 2.2 ([12])**.** *Suppose $n = 3$ and $k_0 + k_1 + k_2 > 1$. Let the function $p(x, u, v, w)$ be continuous, Lipschitz continuous in u, v, w and satisfy the inequality $p(x, u, v, w) \ge m > 0$. Then there exists a constant $\psi = \psi(m, k_0, k_1, k_2)$ such that any maximally extended solution $y(x)$ to equation* (1.1)*, satisfying at some point x_0 the conditions $y(x_0) \ge 0$, $y'(x_0) \ge 0$, $y''(x_0) = y_2 > 0$, has a finite right domain boundary x^* satisfying the estimate*

$$x^* - x_0 < \psi\, y_2^{-\frac{k_0+k_1+k_2-1}{2k_0+k_1+1}}.$$

Consider now the asymptotic behavior of solutions with positive initial data to second- and third order equations. For second order equation the result is also obtained for general form of $p(x, u, v)$ in [11].

Theorem 2.3 ([11])**.** *Suppose $n = 2$ and $k_0 + k_1 > 1$, $k_1 < 2$. Let $p(x, u, v) \equiv p_0 > 0$, and let x^* be a right boundary of the domain of an inextensible solution $y(x)$ to* (1.1) *satisfying at some point x_0 the conditions $y(x_0) > 0$, $y'(x_0) > 0$. Then*

$$y = C(x^* - x)^{-\alpha}(1 + o(1)), \quad x \to x^* - 0,$$

where

$$\alpha = \frac{2 - k_1}{k_0 + k_1 - 1} > 0, \quad C = \Big(\frac{|\alpha|^{1-k_1}|\alpha + 1|}{p_0}\Big)^{\frac{1}{k_0+k_1-1}}.$$

Theorem 2.4. *Suppose $n = 3$ and $k_0+k_1+k_2 > 1$, $k_2 < 1$, $k_1+2k_2 < 3$. Let $p(x, u, v, w) \equiv p_0 > 0$, and let x^* be a right boundary of the domain of an inextensible solution $y(x)$ to* (1.1) *satisfying at some point x_0 the conditions $y(x_0) > 0$, $y'(x_0) > 0$, $y''(x_0) > 0$. Then*

$$y = C(x^* - x)^{-\alpha}(1 + o(1)), \quad x \to x^* - 0,$$

where

$$\alpha = \frac{3 - k_1 - 2k_2}{k_0 + k_1 + k_2 - 1} > 0, \quad C = \Big(\frac{|\alpha|^{1-k_1-k_2}|\alpha + 1|^{1-k_2}|\alpha + 2|}{p_0}\Big)^{\frac{1}{k_0+k_1+k_2-1}}.$$

It turns out that it is possible to generalize the above results for the higher order equation (1.1). Denote

$$K = \sum_{i=0}^{n-1} k_i, \quad \varkappa = \sum_{i=1}^{n-1} i\, k_{n-1-i}.$$

Theorem 2.5. *Suppose $n \ge 2$, $K > 1$. Let the function $p(u_0, u_1, \dots, u_{n-1})$ be continuous in x, Lipschitz continuous in $u_0, \dots, u_{n-1}$ and satisfy the inequality*

$$p(x, y, y', \dots, y^{(n-1)}) \ge m > 0.$$

Then there exists a constant $\xi = \xi(n, m, k_0, \dots, k_{n-1})$ such that any maximally extended solution $y(x)$ to (1.1)*, satisfying at some point x_0 the conditions $y(x_0) > 0$, $y'(x_0) > 0, \dots, y^{(n-2)}(x_0) > 0$, $y^{(n-1)}(x_0) = y_{n-1} > 0$, has a finite right domain boundary x^* satisfying the estimate*

$$x^* - x_0 < \xi\, y_{n-1}^{-\frac{K-1}{\varkappa+1}}.$$

The following theorem states the existence of a solution in the form $y = C(x^* - x)^{-\alpha}$ to equation (1.1) with constant potential $p(x, y, y', \dots, y^{(n-1)}) \equiv (-1)^{n-1}p_0$.

Denote $\overline{\varkappa} = \sum\limits_{i=1}^{n-1} i\, k_i$.

Theorem 2.6. *Let $n \ge 2$, $p_0 > 0$ and $K > 1$. Then equation*

$$y^{(n)} = (-1)^{n-1} p_0 |y|^{k_0} |y'|^{k_1} \cdots |y^{(n-1)}|^{k_{n-1}} \operatorname{sgn}(y\, y' \cdots y^{(n-1)}) \tag{2.1}$$

has a solution $y = C(x^* - x)^{-\alpha}$, where $x^* < \infty$ is the right domain boundary,

$$C = \left(\frac{\prod\limits_{i=0}^{n-1} |\alpha + i|^{1 - \sum\limits_{i+1}^{n-1} k_i}}{p_0} \right)^{\frac{1}{K-1}}, \quad \alpha = \frac{n - \overline{\varkappa}}{K - 1}\,.$$

Note that the higher order Emden–Fowler equation

$$y^{(n)} = p_0 |y|^k \operatorname{sgn} y, \ \ n \ge 2, \ \ k > 1, \ \ p_0 > 0$$

for any $x^* \in \mathbb{R}$ has the solution $y = C(x^* - x)^{-\alpha}$ with

$$\alpha = \frac{n}{k-1}\,, \quad C = \left(\frac{\alpha(\alpha+1) \cdots (\alpha+n-2)(\alpha+n-1)}{p_0} \right)^{\frac{1}{k-1}},$$

which corresponds to the result obtained in Theorem 2.6 with $k_1 = \cdots = k_{n-1} = 0$ (see [1, **5.1**]). The existence of solutions to equation (1.1) which is equivalent to $C(x^* - x)^{-\alpha}$ as $x \to x^* - 0$ in general case is an open problem. For $n = 2$ this problem was solved in [11], and for $n \ge 3$, $k_1 = \cdots = k_{n-1} = 0$ it was solved in [1, Chapter 5] and [4, 6].

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Optimal Control Problems for Systems of Differential Equations with Impulse Action

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The problem of optimal control at a finite time interval for a system of differential equations with impulse action at fixed moments times and also the corresponding averaged system of ordinary differential equations are considered. The existence of optimal control of exact problem and averaged problem is proved, and also it is established that optimal control of averaged task carries out the approximate optimal synthesis of exact problem.

1 Introduction

In this paper, for the system of differential equations with impulse action at fixed moments of time, the problem of optimal control is considered:

$$\begin{gathered}
\dot{x}=\varepsilon\big[A(t,x)+B(t,x)u\big],\ \ t\neq t_i,\ \ i=1,2,\dots,i\Big(\frac{T}{\varepsilon}\Big),\ \ t\in\Big[0,\frac{T}{\varepsilon}\Big),\\
\triangle x\big|_{t=t_i}=\varepsilon I_i(x(t_i),v_i),\ \ i=1,2,\dots,i\Big(\frac{T}{\varepsilon}\Big),\\
x(0,u(0),v_i)=x_0,\ \ t_i<t_{i+1},
\end{gathered}\tag{1.1}$$

where $\varepsilon>0$ is a small parameter, $t\geq 0$, $T>0$ is some constant value, $x\in D$ is a phase n-dimensional vector, D is a region in R^n, $u\in U$ is a vector of control, U is convex and closed set in R^m, $0\in U$, $i(t)$ is the number of pulses on $[0,t)$: $t_1,t_2,\dots,t_n,\dots,t_{i(\frac{T}{\varepsilon})}$, and $t_n\to\infty$, $n\to\infty$; $v_i\in V$, $i=1,2,\dots,i(\frac{T}{\varepsilon})$, are impulse control vectors, V is a closed set in $\mathbb{R}^r$. With respect to the moments of impulsive action, we assume that there exists a constant $\widetilde{C}>0$ such that for $t\geq 0$,

$$i(t)\leq\widetilde{C}t.$$

A is an n-dimensional vector-function, B is an $n\times m$-dimensional matrix, $I_i(x,v)$ is an n-dimensional vector function.

Control $u=u(t)=(u_1(t),u_2(t),\dots,u_m(t))$ and $v=v_i=(v_{i1},v_{i2},\dots,v_{ir})$ will be considered admissible for problem (1.1), if (a1) $u(t) \in L_p(0, \frac{T}{\varepsilon})$ for some $p > 1$;

(a2) $u(t) \in U$ at $t \in [0, \frac{T}{\varepsilon}]$, almost everywhere;

a3) there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ the solution $x(t, u, v)$ of the Cauchy problem (1.1) has defined by $t \in [0, \frac{T}{\varepsilon}]$, where ε_0 is independent of $u(t)$ and v_i;

(a4) $v_i \in V$;

(a5) for each sequence of control vectors $v_i \in V$ there exists the vector $v_0 \in V$ such that $v_i \to v_0$, $i \to \infty$, uniformly for all controls, that is, for arbitrary $\delta > 0$ there is a constant N_0, independent of v_i, v_0 and such that for all $i \geq N_0$ the inequality $|v_i - v_0| < \delta$ is satisfied.

It should be noted that condition a5) is obviously satisfied if there exists a sequence $\{a_i\}$ independent of v_i: $a_i \to 0$, $i \to \infty$, such that $|v_i - v_0| < a_i$.

We denote the set of valid controls by Ω.

By $|\cdot|$ we denote the norm of vector in Euclidean space, and through $\|\cdot\|$ we denote the norm of the matrix consistent with the norm of the vector. In this paper, the averaging method is applied to optimal control problems. The main role here is to justify the closeness of the solutions of the exact and average problems. This type of results for impulse systems was first obtained in [5] and further developed in the works of many scientists and applied to optimal control problems (see, for example, [4], where is comprehensive bibliography).

In works [3, 7, 8], another approach was developed to apply the averaging method to optimal control problems, where the control function was considered a fixed parameter when averaging. This approach had applied to the problems of optimal control of functional-differential equations in [2].

2 Formulation of the problem and the main result

The problem of optimal control to be solved in the work is to find such allowable controls $u(t)$ and v_i that minimize the functional

$$J_\varepsilon(u, v) = \varepsilon \int\limits_0^{\frac{T}{\varepsilon}} [C(t, x) + F(t, u)]\, dt + \varepsilon \sum_{0 \leq t_i < \frac{T}{\varepsilon}} \Psi_i(x(t_i), v_i),$$

here C, F, ψ_i are continuous in the set of variables of function, with $C \geq 0$, F and ψ_i satisfy the conditions:

$F(t, u)$ is defined for $t \geq 0$, $u \in U$, convex on u, and for some $a > 0$:

$$F(t, u) \geq a|u|^p, \quad \psi_i(t, v) \geq a|v|^p,$$

where $p > 1$ from condition $a1)$ and for some $K > 0$ there exists $\varepsilon_0 > 0$ such that $\varepsilon < \varepsilon_0$ the inequality

$$\varepsilon \int\limits_0^{\frac{T}{\varepsilon}} F(t, 0)\, dt \leq K$$

holds.

With respect to system (1.1), we assume that the following conditions are fulfilled:

1.1) there are such $A_0(x)$, $B_0(x)$ and $C_0(x)$, for which uniformly over $x\in D$ the boundaries exist (averaging conditions):

$$\lim_{T\to\infty}\bigg|\frac{1}{T}\int\limits_0^T A(t,x)\,dt-A_0(x)\bigg|=0,$$
$$\lim_{T\to\infty}\bigg|\frac{1}{T}\int\limits_0^T C(t,x)\,dt-C_0(x)\bigg|=0,$$
$$\lim_{T\to\infty}\frac{1}{T}\int\limits_0^T \|B(t,x)-B_0(x)\|^q\,dt=0,$$

where q is determined from the condition $\frac{1}{p}+\frac{1}{q}=1$;

1.2) the vector function $A(t,x)$ and the matrix function $B(t,x)$ are defined, measurable by t for each x, the function $C(t,x)$ is defined and is continuous at $t\ge 0$, $x\in D$;

1.3) the functions $A(t,x)$, $B(t,x)$ and $C(t,x)$ are Lipschitz's functions on x with constant L in domain D;

1.4) the functions $I_i(x,v)$, $\psi_i(x,v)$, $i=1,2,\dots,i(t)$, are continuous on the set of variables;

1.5) the functions $\psi_i(x,v)$, $i=1,2,\dots,i(t)$, are bounded by the constant M at $t\ge 0$, $x\in D$, $v\in V$;

1.6) the functions $I_i(x,v)$, $\psi_i(x,v)$, $i=1,2,\dots,i(t)$, are Lipschitz's functions on x with constant L in the domain D and uniformly continuous on v in the domain of definition;

1.7) for the functions $A(t,x)$, $B(t,x)$, $C(t,x)$ and $I_i(x,v)$, $i=1,2,\dots,i(\frac{T}{\varepsilon})$, the conditions of linear growth are fulfilled, i.e., there is a constant $K>0$ such that for $t\ge 0$ and $x\in D$ the followings inequalities are fulfilled:

$$|A(t,x)|\le K(1+|x|),\ \ \|B(t,x)\|\le K(1+|x|),\ \ |I_i|\le K(1+|x|),\ \ |C(t,x)|\le K(1+|x|).$$

Let the averaging conditions also be satisfied:

1.8) uniformly for $x\in D$, $u\in U$, $v\in V$ there are boundaries:

$$\lim_{s\to\infty}\frac{1}{s}\sum_{0<t_i<s} I_i(x,v)=I_0(x,v),$$
$$\lim_{s\to\infty}\frac{1}{s}\sum_{0<t_i<s} \psi_i(x,v)=\psi_0(x,v).$$

Problem (1.1) on the interval $[0,\frac{T}{\varepsilon}]$ will correspond to the following averaged problem:

$$\begin{gathered}\dot y=\varepsilon\big[A_0(y)+B_0(y)\overline{u}+I_0(y,v_0)\big],\ \ t\in\Big[0,\frac{T}{\varepsilon}\Big),\\ y\big(0,\overline{u}(0),\overline{v}_i(0)\big)=x_0,\end{gathered}\tag{2.1}$$

where $\overline{u}$ is the allowable control of the averaging problem (2.1), that satisfies the same conditions as the allowable control of the exact problem (1.1), and v_0 for each v_i is selected from condition a5).

The set of admissible controls $(u(t), v_0)$ of problem (2.1) is denoted by $\overline{\Omega}$. The quality criterion of the problem of averaging is as follows:

$$\overline{J}_\varepsilon(\overline{u},\overline{v}) = \varepsilon \int_0^{\frac{T}{\varepsilon}} \big[C_0(y(t)) + F(t,\overline{u}) + \psi_0(y(t),v_0)\big]\,dt.$$

Let's denote

$$\begin{aligned} J_\varepsilon^* &= \inf_{(u(t),v_i)\in\Omega} J_\varepsilon(u,v), \\ \overline{J}_\varepsilon^{\,*} &= \inf_{(u(t),v_0)\in\Omega} \overline{J}_\varepsilon(\overline{u},\overline{v}). \end{aligned}$$

The purpose of this work is to prove for the problem of optimal control the following statement: for an arbitrary $\eta > 0$ there is $\varepsilon_0 = \varepsilon_0(\eta)$ such that for $\varepsilon < \varepsilon_0$ the inequality

$$\big|J_\varepsilon^* - J_\varepsilon(\overline{u}^{\,*}, v_0^*)\big| \le \eta$$

holds; $\overline{u}^{\,*}$, v_0^* is the optimal control pair for the problem of averaging, i.e., the optimal control of the problem of averaging is almost optimal for the exact one.

For the averaged system (2.1) we assume that the following condition is fulfilled:

(A) If the control $\overline{u}$ satisfies the estimate

$$\varepsilon \int_0^{\frac{T}{\varepsilon}} |\overline{u}(t)|\,dt \le R,$$

where $R > 0$ does not depend on ε, $\overline{u}$, then there is $\varepsilon_0 = \varepsilon_0(R)$ such that for $0 < \varepsilon < \varepsilon_0$ the solution of the averaged Cauchy problem $y(t,\overline{u},v_0)$ for $t \in [0,\frac{T}{\varepsilon}]$ lies in the region D together with some ρ-neighborhood, and ρ does not depend on ε, $\overline{u}$, v_0.

The following theorem holds.

Theorem. *Under conditions* 1.1)–1.7) *and condition* (A) *there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *the exact and averaged control problems have solutions, and for an arbitrary* $\eta > 0$ *there exists* $\varepsilon_1 = \varepsilon_1(\eta) \le \varepsilon_0$ *such that for* $0 < \varepsilon < \varepsilon_1$ *the inequality*

$$\big|J_\varepsilon^* - J_\varepsilon(\overline{u}^{\,*}, v_0^*)\big| \le \eta$$

is fulfilled, where $(\overline{u}^{\,*}, v_0^*)$ *is the optimal control of the averaging system.*

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On Asymptotic Representations of One Class of Solutions of Second-Order Differential Equations

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Consider the differential equation

$$y'' = f(t, y, y'), \tag{1}$$

where $f : [a, \omega[\, \times \Delta_{Y_0} \times \Delta_{Y_1} \to \mathbf{R}$ is a continuous function, $-\infty < a < \omega \le +\infty$, Δ_{Y_i} $(i \in \{0,1\})$ is a one-side neighborhood of Y_i and Y_i $(i \in \{0,1\})$ is either 0 or $\pm\infty$. We assume that the numbers μ_i $(i = 0, 1)$ given by the formula

$$\mu_i = \begin{cases} 1 & \text{if either } Y_i = +\infty, \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is a right neighborhood of the point } 0, \\ -1 & \text{if either } Y_i = -\infty, \text{ or } Y_i = 0 \text{ and } \Delta_{Y_i} \text{ is a left neighborhood of the point } 0 \end{cases}$$

satisfy the relations

$$\mu_0\mu_1 > 0 \ \text{ for } \ Y_0 = \pm\infty \ \text{ and } \ \mu_0\mu_1 < 0 \ \text{ for } \ Y_0 = 0. \tag{2}$$

Conditions (2) are necessary for the existence of solutions of equation (1) defined in the left neighborhood of ω and satisfying the conditions

$$y^{(i)}(t) \in \Delta_{Y_i} \ \text{ for } \ t \in [t_0, \omega[\,, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1). \tag{3}$$

Among the strictly monotonic, together with the derivatives of the first order, in some left neighborhood of ω of solutions of equation (1) we can single out only solutions admitting either representations of the form

$$y(t) = c_0 + o(1), \quad y(t) = \pi_\omega(t)[c_1 + o(1)] \ \text{ as } \ t \uparrow \omega, \tag{4}$$

where c_0, c_1 are nonzero real constants, or satisfying conditions (3).

The question of whether equation (1) has solutions with representations (4) can be, in general, solved using either for $\omega = +\infty$ a theorem from monograph [3, Ch. II, § 8, p. 207] or for $\omega \le +\infty$ ideas laid down in the work [1].

One of the classes of equation (1) solutions with properties (3) that admits asymptotic representations is the class of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions.

Definition 1. A solution y of equation (1) on the interval $[t_0, \omega[\, \subset [a, \omega[$ is called $P_\omega(Y_0, Y_1, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if, in addition to (3), it satisfies the condition

$$\lim_{t \uparrow \omega} \frac{[y'(t)]^2}{y(t) y''(t)} = \lambda_0.$$

Depending on λ_0 these solutions have different asymptotic properties. For $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ in [2] such ratios
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)y'(t)}{y(t)} = \frac{\lambda_0}{\lambda_0 - 1}\,, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)y''(t)}{y'(t)} = \frac{1}{\lambda_0 - 1}\,,$$
where
$$\pi_\omega(t) = \begin{cases} t & \text{if } \omega = +\infty, \\ t-\omega & \text{if } \omega < +\infty, \end{cases}$$
are established.

Definition 2. We say that a function f satisfies condition $(FN)_{\lambda_0}$ for $\lambda_0 \in \mathbb{R} \setminus \{0,1\}$ if there exist a number $\alpha_0 \in \{-1,1\}$, a continuous function $p : [a,\omega[\to]0,+\infty[$ and twice continuously differentiable function $\varphi_0 : \Delta_{Y_0} \to]0,+\infty[$, satisfying the conditions
$$\varphi_0'(y) \neq 0, \quad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \varphi_0(y) = \varphi_0 \in \{0,+\infty\}, \quad \lim_{\substack{y\to Y_0 \\ y\in\Delta_{Y_0}}} \frac{\varphi_0(y)\varphi_0''(y)}{(\varphi_0'(y))^2} = 1, \tag{5}$$
such that, for arbitrary continuously differentiable functions $z_i : [a,\omega[\to \Delta_{Y_i}$ $(i=0,1)$ satisfying the conditions
$$\lim_{t\uparrow\omega} z_i(t) = Y_i \ \ (i=0,1),$$
$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)z_0'(t)}{z_0(t)} = \frac{\lambda_0}{\lambda_0 - 1}\,, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)z_1'(t)}{z_1(t)} = \frac{1}{\lambda_0 - 1}\,,$$
one has representation
$$f(t,z_0(t),z_1(t)) = \alpha_0 p(t)\varphi_0(z_0(t))[1+o(1)] \ \text{ as } \ t\uparrow\omega. \tag{6}$$

Note that the choice of α_0 and the functions p and φ_0 in Definition 2 depends on the choice of $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$. It is also obvious that the numbers μ_0, μ_1 determine the signs of any $P_\omega(Y_0,Y_1,\lambda_0)$-solution of equation (1) and its derivative in a left neighborhood of ω. Moreover, under condition $(FN)_{\lambda_0}$ sign of second derivative of any $P_\omega(Y_0,Y_1,\lambda_0)$-solution of equation (1) in a left neighborhood of ω coincides with the value α_0. Then taking into account (2), we have
$$\alpha_0\mu_1 > 0 \ \text{ for } \ Y_1 = \pm\infty \ \text{ and } \ \alpha_0\mu_1 < 0 \ \text{ for } \ Y_1 = 0. \tag{7}$$

We choose a number $b \in \Delta_{Y_0}$ such that the inequality
$$|b| < 1 \ \text{ for } \ Y_0 = 0, \quad b > 1 \ \ (b < -1) \ \text{ for } \ Y_0 = +\infty \ \ (Y_0 = -\infty)$$
is respected and put
$$\begin{cases} \Delta_{Y_0}(b) = [b, Y_0[& \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0, \\ \Delta_{Y_0}(b) =]Y_0, b] & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0. \end{cases}$$

Now we introduce auxiliary functions and notation as follows:
$$\Phi : \Delta_{Y_0}(b) \to \mathbb{R}, \quad \Phi(y) = \int_B^y \frac{ds}{\varphi_0(s)}\,, \quad B = \begin{cases} b & \text{if } \displaystyle\int_b^{Y_0} \frac{ds}{\varphi_0(s)} = \pm\infty, \\ Y_0 & \text{if } \displaystyle\int_b^{Y_0} \frac{ds}{\varphi_0(s)} = const, \end{cases}$$

$$Z = \lim_{y\to Y_0} \Phi(y) = \begin{cases} 0 & \text{if } \ B = Y_0, \\ +\infty & \text{if } \ B = b \ \text{ and } \ \mu_0\mu_1 > 0, \\ -\infty & \text{if } \ B = b \ \text{ and } \ \mu_0\mu_1 < 0, \end{cases} \quad \mu_2 = \begin{cases} 1 & \text{if } \ B = b, \\ -1 & \text{if } \ B = Y_0, \end{cases}$$

$$I(t) = \int_A^t \pi_\omega(\tau)p(\tau)\,d\tau, \quad A = \begin{cases} a & \text{if } \displaystyle\int_a^\omega \pi_\omega(\tau)p(\tau)\,d\tau = \pm\infty, \\ \omega & \text{if } \displaystyle\int_a^\omega \pi_\omega(\tau)p(\tau)\,d\tau = const. \end{cases}$$

Theorem 1. *Let $\lambda_0 \in \mathbb{R}\setminus\{0,1\}$ and let the function f satisfy condition $(FN)_{\lambda_0}$. Then, for the existence of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of the differential equation* (1)*, it is necessary that the sign conditions* (2)*,* (7)*,*

$$\alpha_0\mu_0\lambda_0 > 0, \quad \mu_0\mu_1\lambda_0(\lambda_0-1)\pi_\omega(t) > 0, \quad \alpha_0\mu_2(\lambda_0-1)I(t) < 0 \ \text{ for } \ t \in [a,\omega[$$

and

$$\alpha_0(\lambda_0-1)\lim_{t\uparrow\omega} I(t) = Z, \quad \lim_{t\uparrow\omega} \frac{\pi_\omega(t)I'(t)}{I(t)} = \pm\infty, \quad \lim_{t\uparrow\omega} \frac{\alpha_0(\lambda_0-1)\pi_\omega^2(t)p(t)\varphi_0(Y(t))}{Y(t)} = \frac{\lambda_0}{\lambda_0-1}$$

hold, where

$$Y(t) = \Phi^{-1}\big(\alpha_0(\lambda_0-1)I(t)\big).$$

Moreover, each solution of this kind admits the asymptotic representations

$$\frac{y'(t)}{\varphi_0(y(t))} = \alpha_0(\lambda_0-1)\pi_\omega(t)p(t)[1+o(1)], \quad \varphi_0'(y(t)) = -\frac{\lambda_0(1+o(1))}{(\lambda_0-1)I(t)} \ \text{ as } \ t\uparrow\omega.$$

Remark 1. Asymptotic representations of $P_\omega(Y_0, Y_1, \lambda_0)$-solutions of equation (1) can be written explicitly

$$y(t) = Y(t)\Big(1 + \frac{o(1)}{H(t)}\Big), \quad y'(t) = \frac{\lambda_0}{\lambda_0-1}\,\frac{Y(t)}{\pi_\omega(t)}\,(1+o(1)),$$

where

$$H(t) = \frac{Y(t)\varphi'(Y(t))}{\varphi(Y(t))}\,.$$

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On the Existence of Some Solutions of Systems of Ordinary Differential Equations that are Partially Resolved Relatively to the Derivatives with Square Matrix

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Let us consider the system of ordinary differential equations

$$A(z)Y' = B(z)Y + f(z, Y, Y'), \tag{1}$$

where the matrices $A : D_1 \to \mathbb{C}^{p\times p}$, $B : D_{10} \to \mathbb{C}^{p\times p}$, $D_1 = \{z : \ |z| < R_1, \ R_1 > 0\} \subset \mathbb{C}$, $D_{10} = D_1 \setminus \{0\}$, matrix $A = A(z)$ is analytical in the domain D_1, matrix $B = B(z)$ is analytical in the domain D_{10}, $\operatorname{rang} A(z) = p$ in the domain $z \in D_1$, $A^{(-1)}(z)B(z)$ is analytical matrix in the domain D_{10} and has pole of order $d \in \mathbb{N}$ in the point $z = 0$, the vector-function $f : D_1 \times G_1 \times G_2 \to \mathbb{C}^p$, where domains $G_k \subset \mathbb{C}^p$, $0 \in G_k$, $k = 1, 2$, the vector-function $f = f(z, Y, Y')$ is analytical in the domain $D_{10} \times G_{10} \times G_{20}$, $G_{k0} = G_k \setminus \{0\}$, $k = 1, 2$, the decomposition of the vector function $f = f(z, Y, Y')$ to a convergent power series around the point $(0, 0, 0)$ has no free and linear members.

Let us study question on the existence of analytic solutions of the Cauchy problem for system (1) with the initial condition

$$Y \to 0, \ \ z \to 0, \ \ z \in D_{10},$$

and the additional condition

$$Y' \to 0, \ \ z \to 0, \ \ z \in D_{10}.$$

According to these assumptions, system (1) takes the form

$$z^d Y' = \check{P}^{(2)}(z)Y + z^d H^{(2)}(z, Y, Y'), \tag{2}$$

where $\check{P}^{(2)}(z)$ is an analytical matrix in the domain D_1, $H^{(2)} = H^{(2)}(z, Y, Y')$ is an analytical vector-function in the domain $D_1 \times G_1 \times G_2$.

Definition 1. Let's define that the vector-function $z^d H^{(2)}(z, Y, Y')$ has the property V_1 near the point $(0, 0, 0)$ if this neighborhood component vector function $z^d H^{(2)}(z, Y, Y')$ may be decomposed into convergent series form

$$z^d H_j^{(2)}(z, Y, Y') = \sum_{s+|l|+|q|=2}^{\infty} C_{slq}^{(2.j)} z^s Y^l (z^d Y')^q, \ \ j = \overline{1, p},$$

where $C_{slq}^{(2.j)} \in \mathbb{C}$, $j = \overline{1, p}$.

Lemma. *If in system* (2) *vector-function* $z^d H^{(2)}(z, Y, Y')$ *has the property* V_1 *near the point* $(0, 0, 0)$, *then system* (2) *can be uniquely reduced to the system of the type*

$$z^d Y' = P^{(2)}(z)Y + F^{(2)}(z, Y), \tag{3}$$

where $P^{(2)}(z)$ *is an analytical matrix in the domain* $\widetilde{D_1} \subseteq D_1$, $0 \in \widetilde{D_1}$, $F^{(2)} = F^{(2)}(z, Y)$ *is an analytical vector-function in the domain* $\widetilde{D_1} \times \widetilde{G_1} \subseteq D_1 \times G_1$, $(0, 0) \in \widetilde{D_1} \times \widetilde{G_1}$, $F^{(2)}(0, 0) = 0$. *For convenience, we assume that the matrix* $P^{(2)}$ *is analytical in the domain* D_1, *and the vector-function* $F^{(2)}$ *is analytical in the domain* $D_1 \times G_1$.

For arbitrarily fixed $t_1 \in (0, R_1]$, $v_1, v_2 \in \mathbb{R}$, $v_1 < v_2$, introduce a set $\check{I}(t_1) = \{(t,v) \in \mathbb{R}^2 :\ t \in (0,t_1),\ v \in (v_1,v_2)\}$. For $z = z(t,v) = te^{iv}$, the set $\check{I}(t_1) \subset \mathbb{R}^2$ refers to the set $I(t_1) \subset \mathbb{C} :\ I(t_1) = \{z = te^{iv} \in \mathbb{C} :\ t \in (0,t_1),\ v \in (v_1,v_2)\}$.

Definition 2. Let $p, g : \check{I}(t_1) \to [0,+\infty)$. Let's define that the function p has the property Q_1 regarding the function g on the condition $v = v_0 \in (v_1, v_2)$, if the function $p = p(t, v_0)$ is a function of higher order of smallness relative to the function $g = g(t, v_0)$ on the condition $t \to +0$.

Definition 3. Let $p, g : \check{I}(t_1) \to [0,+\infty)$. Let's define that the function p has the property Q_2 regarding the function g on the set $\check{I}(t_1)$, if there exist $C_1 \geq 0$, $C_2 \geq 0$ such that on the set $\check{I}(t_1)$ the inequality

$$C_1 g(t,v) \leq p(t,v) \leq C_2 g(t,v)$$

is satisfied.

Introduce the auxiliary vector function $\varphi(z) = \operatorname{col}(\varphi_1(z), \dots, \varphi_p(z))$, $\varphi : I(t_1) \to \mathbb{C}^p$, and $\psi(t,v) = \operatorname{col}(\psi_1(t,v), \dots, \psi_p(t,v))$, $\psi_j : \check{I}(t_1) \to [0;+\infty)$, $j = \overline{1,p}$, on the condition $z = z(t,v) = te^{iv}$, $\psi_j(t,v) = |\varphi_j(z(t,v))|$, $j = \overline{1,p}$, functions ψ_j, $j = \overline{1,p}$ are really values functions of real variables t, v.

For a fixed $v = v_0$ we introduce

$$\begin{gathered}
Y(z(t,v_0)) = \widetilde{Y}(t), \quad \widetilde{Y}(t) = \widetilde{Y}_1(t) + i\widetilde{Y}_2(t), \\
P^{(2)}(z(t,v_0)) = \|\widetilde{p}^{(2)}_{jk}(t)\|^p_{j,k=1} = \widetilde{P}^{(2)}_1(t) + i\widetilde{P}^{(2)}_2(t), \quad \widetilde{P}^{(2)}_s(t) = \|\widetilde{p}^{(2)}_{jks}(t)\|^p_{j,k=1}, \ \ s = 1,2, \\
F^{(2)}\big(z(t,v_0), Y(z(t,v_0))\big) = \widetilde{F}^{(2)}(t, \widetilde{Y}_1, \widetilde{Y}_2), \\
\widetilde{F}^{(2)}(t, \widetilde{Y}_1, \widetilde{Y}_2) = \operatorname{col}\big(\widetilde{F}^{(2)}_1(t, \widetilde{Y}_1, \widetilde{Y}_2), \dots, \widetilde{F}^{(2)}_p(t, \widetilde{Y}_1, \widetilde{Y}_2)\big), \\
\widetilde{F}^{(2)}_j(t, \widetilde{Y}_1, \widetilde{Y}_2) = \widetilde{F}^{(2)}_{1j}(t, \widetilde{Y}_1, \widetilde{Y}_2) + i\widetilde{F}^{(2)}_{2j}(t, \widetilde{Y}_1, \widetilde{Y}_2), \ \ j = \overline{1,p},
\end{gathered}$$

functions $\widetilde{p}^{(2)}_{jks}(t)$, $j,k = \overline{1,p}$, $s = 1,2$, and vector-functions $\widetilde{Y}_1(t)$, $\widetilde{Y}_2(t)$, $\widetilde{F}^{(2)}_{1j}$, $\widetilde{F}^{(2)}_{2j}$, $j = \overline{1,p}$ are really values functions of real variable t.

For a fixed $t = t_0$ we introduce

$$\begin{gathered}
Y(z(t_0,v)) = \widehat{Y}(v) = \widehat{Y}_1(v) + i\widehat{Y}_2(v), \\
P^{(2)}(z(t_0,v)) = \|\widehat{p}^{(2)}_{jk}(v)\|^p_{j,k=1} = \widehat{P}^{(2)}_1(v) + i\widehat{P}^{(2)}_2(v), \quad \widehat{P}^{(2)}_s(v) = \|\widehat{p}^{(2)}_{jks}(v)\|^p_{j,k=1}, \ \ s = 1,2, \\
F^{(2)}\big(z(t_0,v), Y(z(t_0,v))\big) = \widehat{F}^{(2)}(v, \widehat{Y}_1, \widehat{Y}_2), \\
\widehat{F}^{(2)}(v, \widehat{Y}_1, \widehat{Y}_2) = \operatorname{col}\big(\widehat{F}^{(2)}_1(v, \widehat{Y}_1, \widehat{Y}_2), \dots, \widehat{F}^{(2)}_p(v, \widehat{Y}_1, \widehat{Y}_2)\big), \\
\widehat{F}^{(2)}_j(v, \widehat{Y}_1, \widehat{Y}_2) = \widehat{F}^{(2)}_{1j}(v, \widehat{Y}_1, \widehat{Y}_2) + i\widehat{F}^{(2)}_{2j}(v, \widehat{Y}_1, \widehat{Y}_2), \ \ j = \overline{1,p},
\end{gathered}$$

functions $\widehat{p}^{(2)}_{jks}(v)$, $j,k = \overline{1,p}$, $s = 1,2$, and vector-functions $\widehat{Y}_1$, $\widehat{Y}_2$, $\widehat{F}^{(2)}_{1j}$, $\widehat{F}^{(2)}_{2j}$, $j = \overline{1,p}$ are really values functions of real variable v.

Definition 4. Let's define that the matrix $P^{(2)}(z)$ has the property S_2 regarding the vector-function $\varphi = \varphi(z)$ if the conditions are met:

1) for each $v_0 \in (v_1, v_2)$ functions $t^d(\psi_j(z(t,v)))'_t$ have the property Q_1 regarding the functions $|\widetilde{p}^{(2)}_{jj}(t)|\psi_j(z(t,v))$, $j = \overline{1,p}$, on the condition $v = v_0$;

2) functions $t^{d-1}(\psi_j(t,v))'_v$ have the property Q_2 regarding the functions $|\widehat{p}^{(2)}_{jj}(v)|\psi_j(t,v)$, $j = \overline{1,p}$, on the set $\check{I}(t_2)$ for some $t_2 \in (0,t_1)$;

3) for each $v_0 \in (v_1, v_2)$ functions $|\widetilde{p}^{(2)}_{jk}(t)|\psi_k(t,v)$ have the property Q_1 regarding the functions $t^d(\psi_j(t,v))'_t$, $j,k = \overline{1,p}$, $j \neq k$, on the condition $v = v_0$;

4) functions $|\widehat{p}^{(2)}_{jk}(v)|\psi_k(t,v)$ have the property Q_2 regarding the functions $t^{d-1}(\psi_j(t,v))'_v$, $j,k = \overline{1,p}$, $j \neq k$, on the set $\check{I}(t_2)$ for some $t_2 \in (0,t_1)$.

Let's introduce the sets

$$\widetilde{\Omega}\big(\delta, \varphi(z(t,v_0))\big) = \Big\{(t, \widetilde{Y}_1, \widetilde{Y}_2):\ t \in (0,t_1),\ \widetilde{Y}^2_{1j} + \widetilde{Y}^2_{2j} < \delta^2_j(\psi_j(t,v_0))^2,\ j = \overline{1,p}\Big\},$$

v_0 is fixed on the interval (v_1, v_2),

$$\widehat{\Omega}\big(\tau, \varphi(z(t_0,v))\big) = \Big\{(v, \widehat{Y}_1, \widehat{Y}_2):\ v \in (v_1,v_2),\ \widehat{Y}^2_{1j} + \widehat{Y}^2_{2j} < \tau^2_j(\psi_j(t_0,v))^2,\ j = \overline{1,p}\Big\},$$

t_0 is fixed on the interval $(0,t_1)$, where $\delta = (\delta_1, \dots, \delta_p)$, $\tau = (\tau_1, \dots, \tau_p)$, $\delta_j, \tau_j \in \mathbb{R} \setminus \{0\}$, $j = (1,p)$.

Definition 5. Let's define that the vector-function $F^{(2)} = F^{(2)}(z,Y)$ has the property M_2 regarding the vector-function $\varphi = \varphi(z)$ if the conditions are met:

1) for each $v_0 \in (v_1, v_2)$ on the condition $(t, \widetilde{Y}_1, \widetilde{Y}_2) \in \widetilde{\Omega}(\delta, \varphi(z(t,v_0)))$ functions $\widetilde{F}^{(2)}_{kj} = \widetilde{F}^{(2)}_{kj}(t, \widetilde{Y}_1, \widetilde{Y}_2)$ have the property Q_1 regarding the functions $|\widetilde{p}^{(2)}_{jj}(t)|\psi_j(t,v)$, $j = \overline{1,p}$, $k = 1,2$, on the condition $v = v_0$;

2) for each $(v, \widehat{Y}_1, \widehat{Y}_2) \in \widehat{\Omega}(\tau, \varphi(z(t_0,v)))$ functions $\widehat{F}^{(2)}_{kj} = \widehat{F}^{(2)}_{kj}(v, \widehat{Y}_1, \widehat{Y}_2)$ have the property Q_2 regarding the function $|\widehat{p}^{(2)}_{jj}(v)|\psi_j(t,v))$, $j = \overline{1,p}$, $k = 1,2$, on the set $\check{I}(t_2)$ for some $t_2 \in (0,t_1)$.

Let's introduce domains $\Lambda^{(2)}_{+.k}(t_2)$, $k \in \{+,-\}$, which are defined as

$$\begin{aligned}\Lambda^{(2)}_{+.+}(t_2) = \Big\{(t,v) : \cos\big((d-1)v - \widetilde{\alpha}^{(2)}_{jj}(t)\big) > 0,\ \sin\big((d-1)v - \widehat{\alpha}^{(2)}_{jj}(v)\big) > 0,&\\ j = \overline{1,p},\ t \in (0,t_2),\ v \in (v_1,v_2)\Big\},&\\ \Lambda^{(2)}_{+.-}(t_2) = \Big\{(t,v) :\ \cos\big((d-1)v - \widetilde{\alpha}^{(2)}_{jj}(t)\big) > 0,\ \sin\big((d-1)v - \widehat{\alpha}^{(2)}_{jj}(v)\big) < 0,&\\ j = \overline{1,p},\ t \in (0,t_2),\ v \in (v_1,v_2)\Big\},&\end{aligned}$$

where functions $\widetilde{\alpha}^{(2)}_{jj}(t)$, $\widehat{\alpha}^{(2)}_{jj}(v)$, $j = \overline{1,p}$, are defined through the corresponding diagonal elements of the matrices $\widetilde{P}^{(2)}_q$, $\widehat{P}^{(2)}_q$, $q = 1,2$.

Definition 6. Let's define that system (3) belongs to the class $C^{(2)}_{+.k}, k \in \{+,-\}$ if matrices $P^{(2)}(z) = P^{(2)}(te^{iv})$ are such that $(t,v) \in \Lambda^{(2)}_{+.k}(t_2)$, $k \in \{+,-\}$.

Let's introduce domains $G^{(2)}_{+.k}(t_2) = \{z = z(t,v):\ 0 < |z| < t_2,\ (t,v) \in \Lambda^{(2)}_{+.k}(t_2)\}$, $k \in \{+,-\}$.

Theorem. *Let $A(z)$ be an analytical matrix in the domain D_1 and $rangA(z) = p$ on the condition $z \in D_1$. Let system* (1) *may lead to the appearance* (2)*. The vector-function $z^d H^{(2)}(z,Y,Y')$ has the property V_1 near the point $(0,0,0)$. Moreover, the following conditions are met for system* (3)*:*

1) *the matrix $P^{(2)}(z)$ is analytical in the domain D_1 and has the property S_2 regarding the vector-function $\varphi = \varphi(z)$;*

2) *the vector-function* $F^{(2)} = F^{(2)}(z, Y)$ *is analytical in the domain* $D_1 \times G_1$, $F^{(2)}(0, 0) = 0$ *and has the property* M_2 *regarding the vector-function* $\varphi = \varphi(z)$;

3) *system* (3) *belongs to one of the classes* $C^{(2)}_{+.k}$, $k \in \{+, -\}$.

Then for each $k \in \{+, -\}$ *and for some* $t^* \in (0, t_2)$ *there are solutions of system* (1) $Y = Y(z)$, *which satisfy the initial conditions* $Y(z_0) = Y_0$ *for* $z_0 \in G^{(2)}_{+.k}(t^*)$, $Y_0 \in \{Y : \ |Y_j(z_0)| < \delta_j |\varphi_j(z_0)|, \ \delta_j > 0, \ j = \overline{1, p}\}$, *that are analytical in the domain* $G^{(2)}_{+.k}(t^*)$ *and for these solutions in this particular domain the estimates are fair:*

$$|Y_j(z)|^2 < \delta_j^2 |\varphi_j(z)|^2, \ \ j = \overline{1, p}.$$

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On Positive Periodic Solutions to Parameter-Dependent Second-Order Differential Equations with a Singularity

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We are interested in the existence and non-existence of a **positive** solution to the periodic boundary value problem

$$\boxed{u'' = p(t)u - \frac{h(t)}{u^\lambda} + \mu f(t); \quad u(0) = u(\omega), \quad u'(0) = u'(\omega).} \tag{0.1}$$

Here, $p, h, f \in L([0, \omega])$,

$$h(t) \geq 0 \ \text{ for a.e. } \ t \in [0, \omega], \quad h(t) \not\equiv 0,$$

$\lambda > 0$, and a parameter $\mu \in \mathbb{R}$. By a *solution* to problem (0.1), as usual, we understand a function $u\colon [0, \omega] \to]0, \infty[$ which is absolutely continuous together with its first derivative, satisfies the given equation almost everywhere, and meets periodic conditions.

Definition 0.1. We say that the function $p \in L([0, \omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0, \omega])$ satisfying

$$u''(t) \geq p(t)u(t) \ \text{ for a.e. } \ t \in [0, \omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \geq 0 \ \text{ for } \ t \in [0, \omega] \quad \big(\text{resp. } \ u(t) \leq 0 \ \text{ for } \ t \in [0, \omega]\big)$$

is fulfilled.

Definition 0.2. We say that the function $p \in L([0, \omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega) \tag{0.2}$$

has a positive solution.

For the cases $p \in \mathcal{V}^-(\omega)$, $p \in \mathcal{V}_0(\omega)$, and $p \in \mathcal{V}^+(\omega)$, we provide some results concerning the existence or non-existence of solutions to problem (0.1) depending on the choice of a parameter μ.

1 The case $p \in \mathcal{V}^-(\omega)$

Theorem 1.1. *Let $p \in \mathcal{V}^-(\omega)$. Then, there exist $-\infty \le \mu_* < 0$ and $0 < \mu^* \le +\infty$ such that*

- *for any $\mu \in]\mu_*, \mu^*[$, problem (0.1) has a unique solution,*
- *if $\mu_* > -\infty$, then, for any $\mu \le \mu_*$, problem (0.1) has no solution,*
- *if $\mu^* < +\infty$, then, for any $\mu \ge \mu^*$, problem (0.1) has no solution.*

2 The case $p \in \mathcal{V}_0(\omega)$

Theorem 2.1. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\int\limits_0^\omega f(t)u_0(t)\,\mathrm{d}t > 0,$$

where u_0 is a solution to problem (0.2). Then, there exists $0 < \mu^* \le +\infty$ such that

- *for any $\mu \le 0$, problem (0.1) has no positive solution,*
- *for any $\mu \in]0, \mu^*[$, problem (0.1) has a unique solution,*
- *if $\mu^* < +\infty$, then, for any $\mu \ge \mu^*$, problem (0.1) has no solution.*

From Theorem 2.1, we derive immediately the following result.

Theorem 2.2. *Let $p \in \mathcal{V}_0(\omega)$ and*

$$\int\limits_0^\omega f(t)u_0(t)\,\mathrm{d}t < 0,$$

where u_0 is a solution to problem (0.2). Then, there exists $-\infty \le \mu_* < 0$ such that

- *if $\mu_* > -\infty$, then, for any $\mu \le \mu_*$, problem (0.1) has no solution,*
- *for any $\mu \in]\mu_*, 0[$, problem (0.1) has a unique solution,*
- *for any $\mu \ge 0$, problem (0.1) has no positive solution.*

3 The case $p \in \mathcal{V}^+(\omega)$

Remark 3.1. In [1, Theorem 16.4], it is shown that, if $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and

$$\int\limits_0^\omega [f(t)]_+\,\mathrm{d}t > \nu^*(p)\bigg(\frac{\omega}{4}\int\limits_0^\omega [p(s)]_+\,\mathrm{d}s\bigg)\int\limits_0^\omega [f(t)]_-\,\mathrm{d}t, \tag{3.1}$$

where the number $\nu^*(p)$ depends only on p (see [1, formula (6.22)]), then the linear periodic problem

$$u'' = p(t)u + f(t); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

possesses a unique solution u which is positive.

Theorem 3.1. *Let $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ and (3.1) hold. Then, there exists $0 \le \mu_* < \infty$ such that*

- *for any* $\mu > \mu_*$*, problem* (0.1) *has a solution,*
- *if* $\mu_* > 0$*, then, for any* $\mu < \mu_*$*, problem* (0.1) *has no solution,*
- *if* $\mu_* = 0$*, then, for any* $\mu \leq 0$*, problem* (0.1) *has no solution.*

From Theorem 3.1, we derive immediately the following result.

Theorem 3.2. *Let* $p \in \operatorname{Int} \mathcal{V}^+(\omega)$ *and*

$$\int_0^\omega [f(t)]_- \, \mathrm{d}t > \nu^*(p) \biggl(\frac{\omega}{4} \int_0^\omega [p(s)]_+ \, \mathrm{d}s \biggr) \int_0^\omega [f(t)]_+ \, \mathrm{d}t,$$

Then, there exists $-\infty < \mu^* \leq 0$ *such that*

- *for any* $\mu < \mu^*$*, problem* (0.1) *has a solution,*
- *if* $\mu^* < 0$*, then, for any* $\mu > \mu^*$*, problem* (0.1) *has no solution,*
- *if* $\mu^* = 0$*, then, for any* $\mu \geq 0$*, problem* (0.1) *has no solution.*

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How to Assign Lyapunov Spectra for Completely Controllable Systems

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Consider a linear controllable differential system

$$\dot{x} = A(t)x + B(t)u, \ \ x \in \mathbb{R}^n, \ \ u \in \mathbb{R}^m, \ \ t \geq 0, \tag{1}$$

with piecewise continuous and bounded coefficient matrices A and B. We denote the Cauchy matrix of the corresponding free system

$$\dot{x} = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

by $X(t,s)$, where $t,s \geq 0$, and the Lyapunov exponents of (2) by $\lambda_k(A)$, $k = 1, \dots, n$.

Suppose that the control u is formed as a linear feedback $u = U(t)x$, where the matrix U is also piecewise continuous and bounded. Then the closed-loop system

$$\dot{y} = \big(A(t) + B(t)U(t)\big)y, \ \ y \in \mathbb{R}^n, \ \ t \geq 0, \tag{3}$$

should be treated as a linear differential system with bounded piecewise continuous coefficients. So, all Lyapunov invariants (i.e. invariants of Lyapunov transformations) including Lyapunov exponents $\lambda_k(A + BU)$, $k = 1, \dots, n$, are defined for system (3). Further we assume that the Lyapunov exponents of each system (both (1) and (3)) are arrayed in increasing order as follows

$$\lambda_1(A) \leq \cdots \leq \lambda_n(A)$$

and, respectively,

$$\lambda_1(A + BU) \leq \cdots \leq \lambda_n(A + BU).$$

According to classical definition due to Kalman [1] system (1) is said to be uniformly completely controllable if there exist positive real numbers ϑ and α_i, $i = 1, \dots, 4$, such that for all $\tau \in \mathbb{R}$ the inequalities

$$\alpha_1 I \leqslant W(\tau, \tau + \vartheta) \leqslant \alpha_2 I, \tag{4}$$

$$\alpha_3 I \leqslant \widehat{W}(\tau, \tau + \vartheta) \leqslant \alpha_4 I \tag{5}$$

hold. Here the controllability matrix W (Kalman matrix) is given by

$$W(t_0, t_1) = \int\limits_{t_0}^{t_1} X(t_0, s)B(s)B^T(s)X^T(t_0, s)\, ds,$$

$\widehat{W}(t_0,t_1)=X(t_1,t_0)W(t_0,t_1)X^T(t_1,t_0)$, and $I\in\mathbb{R}^{n\times n}$ is the identity matrix.

System (1) is said to be ϑ-uniformly completely controllable for some given $\vartheta>0$ if the conditions of the above definition are satisfied for this value of ϑ.

The matrix inequalities (4) and (5) should be understood as a conditional notation of inequalities between corresponding quadratic forms. Namely, conditions (4) and (5) mean that the inequalities

$$\alpha_1\|h\|^2\leqslant h^TW(\tau,\tau+\vartheta)h=\int\limits_{\tau}^{\tau+\vartheta}\big\|h^TX(\tau,s)B(s)\big\|^2\,ds\leqslant\alpha_2\|h\|^2,$$

$$\alpha_3\|h\|^2\leqslant h^T\widehat{W}(\tau,\tau+\vartheta)h=\int\limits_{\tau}^{\tau+\vartheta}\big\|h^TX(\tau+\vartheta,s)B(s)\big\|^2\,ds\leqslant\alpha_4\|h\|^2$$

are valid for all $h\in\mathbb{R}^n$.

Since coefficients of system (1) are piecewise continuous and bounded, we can equivalently reformulate Kalman's definition in a somewhat simpler way as follows. System (1) is uniformly completely controllable if there exist positive real numbers ϑ and α such that for all $\tau\in\mathbb{R}$ the inequalities

$$W(\tau,\tau+\vartheta)\geqslant\alpha I$$

hold. In this case an alternative form for definition of uniform complete controllability was given by E. L. Tonkov in [3]. We say that system (1) is ϑ-uniformly completely controllable if there exists a number $l>0$ such that for any state x_0 and each segment $[\tau,\tau+\vartheta]$ there exist a control u ensuring the transfer of system (1) from x_0 to 0 on this segment and satisfying the condition $\|u(t)\|\leqslant l\|x_0\|$ for all $t\in[\tau,\tau+\vartheta]$. For systems with piecewise continuous and bounded coefficients both of the definitions are equivalent.

The following result is well known in the theory of control of asymptotic invariants [2, p. 337].

Theorem 1. *If system* (1) *is uniformly completely controllable and the matrix B is piecewise uniformly continuous (i.e. B can be represented as a sum of uniformly continuous and piecewise constant matrices), then the Lyapunov exponents of system* (3) *are globally controllable.*

Recall that the Lyapunov exponents of system (3) are said to be globally controllable if for any given $\mu_k\in\mathbb{R}$, $k=1,\dots,n$, such that $\mu_1\le\cdots\le\mu_n$ there exists a bounded piecewise continuous feedback matrix U such that the equalities $\lambda_k(A+BU)=\mu_k$ are valid for all $k=1,\dots,n$. These facts motivate us to refer to the matrix U as a matrix control.

If system (1) is completely controllable, but is not uniformly completely controllable, then for any initial time $t_0\ge0$ there exists a $t_1(t_0)>t_0$ such that for any state x_0 of system (1) one can find a control u steering the system from x_0 to zero state on the interval $[t_0,t_1(t_0)]$. Note that in this case no condition is posed on the norm of the control function u and the length of the segment $[t_0,t_1(t_0)]$ is allowed to grow indefinitely when the starting point t_0 moves away from zero.

Proving Theorem 1 we evaluate the Lyapunov exponents of system (3) along the sequence $k\vartheta$, $k\in\mathbb{N}$, and ensure boundedness of matrix control using the property provided by definition of uniform complete controllability in the form due to Tonkov. So we have to conclude that approaches used to prove Theorem 1 are not suitable to solve the Lyapunov spectra assignment problem when the original system (1) is not uniformly completely controllable.

The most natural way to overcome this problem is to use more rapidly growing sequences. In this case, as before, we retain the ability to construct the required control by the aid of the matrix system

$$\dot X=A(t)X+B(t)V,\ \ X\in\mathbb{R}^{n\times n},\ \ V\in\mathbb{R}^{m\times n},\ \ t\ge0,$$

corresponding to system (1).

To implement this idea we introduce two functions describing some controllability properties of system (1). For each $t \in \mathbb{R}$, $t > 0$, by $T(t)$ we denote the exact lower bound of the set of all $\tau \in \mathbb{R}$, $\tau > t$, such that system (1) is completely controllable on $[t, \tau]$.

For each $t, s \in \mathbb{R}$, where $t > T(s) \geqslant s$, by $\Gamma(t, s)$ we denote the exact lower bound of the set of all numbers γ such that for any state x_0 of system (1) there exists a control u steering the system from x_0 to 0 on the segment $[s, t]$ and satisfying the estimate $\|u(\tau)\| \leqslant \gamma \|x_0\|$ for all $\tau \in [s, t]$. Additionally, we assume $\Gamma(t, s) = +\infty$ when $t \leqslant T(s)$.

We say that the sequence t_k, $k \in \mathbb{N}$, satisfies the slow growth condition if $\lim\limits_{k \to \infty} \frac{t_{k+1}}{t_k} = 1$.

Example. Consider a scalar system

$$\dot{x} = b(t)u, \quad x \in \mathbb{R}, \quad u \in \mathbb{R}, \quad t \geq 0, \tag{6}$$

having the form (1) with zero 1×1-matrix A. Let $t_k > 0$, $k \in \mathbb{N}$ be a monotonically increasing to $+\infty$ sequence and s_k, $k \in \mathbb{N}$ be a sequence satisfying the condition $t_{k-1} < s_k < t_k$ for all $k \geqslant 2$.

Let us define a scalar function b as follows: $b(t) = 1$, $t \in]s_k, t_k]$, and $b(t) = 0$ for all other $t \geq 0$. It can be easily proved that system (6) is completely controllable and is not uniformly completely controllable if the sequence $s_k - t_{k-1}$ is unbounded. By direct calculation we assert that $T(t) = s_k$ for $t \in [t_{k-1}, s_k]$ and $T(t) = t$ for $t \in [s_k, t_k]$. Moreover, for $s \in [t_{k-1}, s_k]$, $t \in]s_k, t_k]$ we have $\Gamma(t, s) = (t - s_k)^{-1}$.

The following statements are valid.

(i) If $k/t_k \to 0$ as $k \to \infty$ and the sequence $t_k - s_k$ is bounded, then the Lyapunov exponent of system (6) equals to zero whatever control we choose.

(ii) If $k/t_k \to 0$ as $k \to \infty$ and the sequence s_k satisfies the condition $s_k = \mu t_{k-1} + (1 - \mu) t_k$ with some $\mu \in]0, 1[$, then choosing an appropriate control u we can prescribe any value for the exponent of system (6). Note that we can choose u to be a constant.

Theorem 2. *Suppose that system* (1) *is completely controllable and the matrix B is piecewise uniformly continuous. If there exists a monotonically increasing to $+\infty$ sequence t_k, $k \in \mathbb{N}$, of positive real numbers satisfying the slow growth condition and such that for some $\alpha > 0$ the inequalities*

$$\Gamma(t_{k+1}, t_k) \leqslant \alpha (t_{k+1} - t_k)^{-1},$$

are valid for all $k \in \mathbb{N}$, then the Lyapunov exponents of system (3) *are globally controllable.*

Remark. If the sequence t_k does not satisfy the slow growth condition, then our ability to assign the Lyapunov spectrum of system (3) depends on finer asymptotic properties of free system (2).

Corollary 1. *Suppose that system* (1) *is completely controllable and the matrix B is piecewise uniformly continuous. If there exists a monotonically increasing to $+\infty$ sequence t_k, $k \in \mathbb{N}$, of positive real numbers satisfying the slow growth condition such that for some $\gamma > 0$ the inequalities*

$$W(t_k, t_{k+1}) \geqslant \gamma (t_{k+1} - t_k) I,$$

are valid for all $k \in \mathbb{N}$, then the Lyapunov exponents of system (3) *are globally controllable.*

To prove Corollary 1 we use the standard Kalman controls [1] existing on each segment where some controllable system is completely controllable. These control functions are useless for immediate constructing of necessary matrix controls, but their norms satisfy an appropriate estimate to apply Theorem 2.

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An Approach to Two-Sided Pointwise Estimating Solutions of Linear Boundary Value Problems with an Uncertainty

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1 Introduction

In this communication we are concerned with the question on estimates of solutions to the general linear boundary value problem

$$\mathcal{L}x = f, \quad \ell x = 0, \tag{1.1}$$

in the case of an uncertainty with respect to the right-hand side of the functional differential system $\mathcal{L}x = f$. The linear operator $\mathcal{L} : AC^n \to L^n$ is assumed to be bounded and to have a Fredholm principal part [1, pp. 7, 42]; $\ell : AC^n \to R^n$ is linear bounded vector-functional with linearly independent components, $f \in L^n$. Here L^n is the space of summable functions $f : [0,T] \to R^n$, AC^n is the space of absolutely continuous functions $x : [0,T] \to R^n$. The spaces L^n , AC^n are assumed to be equipped with natural norms.

The right-hand side f is assumed to be known with an uncertainty, namely, it is given only that its values are constrained by the inequalities

$$\Lambda \cdot f(t) \leq \gamma, \quad t \in [0,T], \tag{1.2}$$

with a constant $(N \times n)$-matrix Λ and $\gamma \in R^N$. We assume that the solutions set, V, to the system $\Lambda v \leq \gamma$ is nonempty and bounded.

The question we discuss here is one of the two-sided estimates of any solution to (1.1) at a fixed point, say, $\tau \in [0,T]$:

$$q^1 \leq x(\tau) \leq q^2. \tag{1.3}$$

The consideration is based on the Green operator

$$G : L^n \to AC^n, \quad (Gf)(t) = \int\limits_0^T G(t,s) f(s)\, ds \tag{1.4}$$

of (1.1). For the existence of G and the integral representation of it we refer to [1, pp. 46–49]. Recall that the matrix kernel $G(t,s)$ is called the Green matrix to (1.1). We can understand the estimate (1.3) as an external estimate of the range to G over all f's such that $f(t) \in V$ for almost all $t \in [0,T]$.

2 The two-sided estimation of solutions

We will use the following notation.

Fix a τ from the segment $[0,T]$. Let $(e_1,\dots,e_n)$ be the canonical basis in R^n, thus e_i has 1 as the i-th component and zero as the rest ones. Introduce the vector $e_i^k=(-1)^{k-1}e_i$, $i=1,\dots,n$, $k=1,2$. Define $G_i^k(s)=(e_i^k)'\cdot G(\tau,s)$, where $(\,\cdot\,)'$ stands for transposition.

Denote by $w_i^k(s)$ the solution of the problem $G_i^k(s)\cdot v\to\max$, $v\in V$. Fix a collection of s_j, $j=0,\dots,\mu$, $0=s_0<s_1<\dots<s_\mu=T$, and define $\widetilde{w}_i^k(s)=\sum\limits_{j=1}^{\mu}\chi_{[s_{j-1},s_j)}(s)w_i^k(s_j)$, where $\chi_A(s)$ is the characteristic function of a set $A\subset R$.

Theorem. *Let $G(\tau,s)$ be piecewise continuous in s on $[0,T]$ and nonnegative δ_i^k, $i=1,2,\dots,n$, $k=1,2$ be such that the inequalities*

$$\int\limits_0^T G_i^k(\tau,s)w_i^k(s)\,ds\le\int\limits_0^T G_i^k(\tau,s)\widetilde{w}_i^k(s)\,ds+\delta_i^k=q_i^k,\ \ i=1,2,\dots,n,\ k=1,2,$$

hold. Then, for any f constrained by (1.2)*, there take place the estimates of $x(\tau)$:*

$$(e_i^k)'x(\tau)\le q_i^k,\ \ i=1,2,\dots,n,\ k=1,2.$$

Example. Let us consider the system (see [3])

$$\begin{aligned}\dot{x}_1(t)&=x_2(t-1)+f_1(t),\\ \dot{x}_2(t)&=-x_2(t)+f_2(t),\end{aligned}\quad t\in[0,3],\tag{2.1}$$

where $x_2(s)=0$ if $s<0$. Set up the boundary conditions by the equality

$$\ell x\equiv x(3)-x(0)=0.\tag{2.2}$$

As for the right-hand side f, the information about it is confined only to the inequalities

$$\begin{gathered}-0.25\le f_1(t)\le-0.15,\quad 0.1\le f_2(t)\le 0.5,\\ 0.4f_1(t)-0.1f_2(t)\ge-0.11,\quad 0.4f_1(t)+0.1f_2(t)\le-0.05.\end{gathered}\tag{2.3}$$

The boundary value problem (2.1), (2.2) is iniquely solvable since (2.1) has the fundamental matrix

$$X(t)=\begin{pmatrix}1 & \chi_{[1,3]}(t)(1-e^{1-t})\\ 0 & e^{-t}\end{pmatrix},$$

and

$$\ell X=\begin{pmatrix}-1 & 1-e^{-2}\\ 0 & e^{-3}-2\end{pmatrix},$$

with $\det\ell X=2-e^{-3}$.

Put $\tau=2$ and obtain the estimate of $x(2)$ that holds for any f constrained by (2.3).

Having in mind the Cauchy matrix to the system (2.1) constructed in [3], we can construct the Green matrix $G(t,s)$ to (2.1), (2.2). For purpose of estimating $x(2)$, it suffices to use the section

$G(2,s)$. Let us give its description component by component:

$$G_{11}(2,s) = \begin{cases} 2, \ s \in [0,2], \\ 1, \ s \in (2,3]; \end{cases}$$

$$G_{12}(2,s) = \begin{cases} -\dfrac{e^{-3} - 2 + 2e^{s-2} - 2e^{s-3} + e^{s-4}}{2 - e^{-3}} + 1 - e^{s-1}, & s \in [0,2], \\ -\dfrac{e^{-3} - 2 + 2e^{s-2} - 2e^{s-3} + e^{s-4}}{2 - e^{-3}}\,, & s \in (2,3]; \end{cases}$$

$$G_{22}(2,s) = 0; \quad G_{11}(2,s) = \begin{cases} \dfrac{e^{s-5}}{2 - e^{-3}} + e^{s-2}, & s \in [0,2], \\ \dfrac{e^{s-5}}{2 - e^{-3}}\,, & s \in (2,3]. \end{cases}$$

In this case Theorem gives the following estimates:

$$-1.19 \le x_1(2) \le -0.52; \quad 0.1 \le x_2(2) \le 0.28.$$

To illustrate the interrelation between the rigidity of constraints and the size of the values set to $x(2)$, we note that in the case of $-0.01 \le f_1(t) \le 0.01$, $-0.01 \le f_2(t) \le 0.01$, we obtain $-0.07 \le x_1(2) \le 0.07$, $-0.01 \le x_2(2) \le 0.01$. Clear, it depends on the Green operator property, but the approach we discuss opens a way to take into account specific properties of solution components in contrary to the estimates in terms of the norms introduced into the corresponding functional spaces.

In conclusion we refer to the papers [2–7] where different aspects of the problem on enclosing solutions to various classes of dynamic systems are presented and useful references can be found.

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Disconjugacy for the Fourth Order Ordinary Differential Equations

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In the paper, we study the question of the disconjugacy on the interval $I := [a,b] \subset [0,+\infty[$ of the fourth order linear ordinary differential equation

$$u^{(4)}(t) = p(t)u(t), \tag{0.1}$$

where $p : I \to \mathbb{R}$ is Lebesgue integrable function.

Throughout the paper we use the following notations.

$\mathbb{R} =]-\infty,+\infty[\,$, $\mathbb{R}^+ =]0,+\infty[\,$, $\mathbb{R}_0^+ = [0,+\infty[\,$, $\mathbb{R}_0^- = \mathbb{R} \setminus \mathbb{R}^+$, $\mathbb{R}^- = \mathbb{R} \setminus \mathbb{R}_0^+$.

$C(I;\mathbb{R})$ is the Banach space of continuous functions $u : I \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in I\}$.

$\widetilde{C}^3(I;\mathbb{R})$ is the set of functions $u : I \to \mathbb{R}$ which are absolutely continuous together with their third derivatives.

$L(I;\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : I \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\,ds$.

For arbitrary $x, y \in L(I;\mathbb{R})$, the notation

$$x(t) \preccurlyeq y(t) \ \ \big(x(t) \succcurlyeq y(t)\big) \ \text{ for } \ t \in I,$$

means that $x \le y$ ($x \ge y$) and $x \ne y$. Also we use the notations $[x]_\pm = (|x| \pm x)/2$.

By a solution of equation (0.1) we understand a function $u \in \widetilde{C}^3(I;\mathbb{R})$, which satisfies equation (0.1) a.e. on I.

Definition 0.1. Equation (0.1) is said to be disconjugate (non oscillatory) on I, if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity. Otherwise, we say that equation (0.1) is oscillatory on I.

Definition 0.2. We say that $p \in D_+(I)$ if $p \in L(I;\mathbb{R}_0^+)$, and equation (0.1), under the boundary conditions

$$u^{(i)}(a) = 0, \quad u^{(i)}(b) = 0 \ \ (i = 0,1), \tag{0.1$_2$}$$

has a solution u such that

$$u(t) > 0 \ \text{ for } \ t \in \,]a,b[\,. \tag{0.2}$$

Definition 0.3. We say that $p \in D_-(I)$ if $p \in L(I;\mathbb{R}_0^-)$, and equation (0.1) under the boundary conditions

$$u^{(i)}(a) = 0 \ \ (i = 0,1,2), \quad u(b) = 0 \tag{0.2$_3$}$$

has a solution u such that inequality (0.2) holds.

1 Main results

1.1 Disconjugacy of equation (0.1) with non-negative coefficient

First we consider equation (0.1) when the coefficient p is non-negative. In this case the following propositions are valid.

Theorem 1.1. *Let $p \in L(I;\mathbb{R}_0^+)$. Then equation* (0.1) *is disconjugate on I iff there exists $p^* \in D_+(I)$ such that*

$$p(t) \preccurlyeq p^*(t) \;\; \text{for} \;\; t \in I. \tag{1.1}$$

Remark 1.1. From Theorem 1.1 it is clear that the structure of the set $D_+(I)$ is such that if $x, y \in D_+(I)$, then none of the inequalities $x \preccurlyeq y$ and $y \preccurlyeq x$ holds.

The following corollary shows us that for an arbitrary $p^* \in D_+(I)$, inequality (1.1) is optimal in some sense.

Corollary 1.1. *Let $p^* \in D_+(I)$ and*

$$p(t) \geq p^*(t) \;\; \text{for} \;\; t \in I. \tag{1.2}$$

Then equation (0.1) *is oscillatory on I.*

Let $\lambda_1 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = \lambda^4 u(t), \quad u^{(i)}(0) = 0, \quad u^{(i)}(1) = 0 \;\; (i = 0, 1). \tag{1.3}$$

Then from Theorem 1.1 and Corollary 1.1 we obtain

Corollary 1.2. *Equation* (0.1) *is disconjugate on I if*

$$0 \leq p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \;\; \text{for} \;\; t \in I,$$

and is oscillatory on I if

$$p(t) \geq \frac{\lambda_1^4}{(b-a)^4} \;\; \text{for} \;\; t \in I.$$

Remark 1.2. It is well known that the first eigenvalue λ_1 of problem (1.3), is the first positive root of the equation $\cos\lambda \cdot \cosh\lambda = 1$, and $\lambda_1 \approx 4.73004$ (see [2,5]). Also, in Theorem 3.1 of paper [5] it was proved that the equation $u^{(4)} = \lambda^4 u$ is disconjugate on $[0,1]$ if $0 \leq \lambda < \lambda_1$.

1.2 Disconjugacy of equation (0.1) with non-positive coefficient

Now we consider equation (0.1) with the non-positive coefficient p, for which the following propositions are valid.

Theorem 1.2. *Let $p \in L(I;\mathbb{R}_0^-)$. Then equation* (0.1) *is disconjugate on I iff there exists $p_* \in D_-(I)$ such that*

$$p(t) \succcurlyeq p_*(t) \;\; \text{for} \;\; t \in I. \tag{1.4}$$

Remark 1.3. From Theorem 1.2 it is clear that the structure of the set $D_-(I)$ is such that if $x, y \in D_-(I)$, then none of the inequalities $x \preccurlyeq y$ and $y \preccurlyeq x$ holds.

The following corollary shows us that for an arbitrary $p^* \in D_-(I)$, inequality (1.4) is optimal in some sense.

Corollary 1.3. *Let $p_* \in D_-(I)$, and*

$$p(t) \le p_*(t) \ \ \text{for} \ \ t \in I. \tag{1.5}$$

Then equation (0.1) *is oscillatory on I.*

Let $\lambda_2 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = -\lambda^4 u(t), \quad u^{(i)}(0) = 0 \ \ (i = 0,1,2), \quad u(1) = 0. \tag{1.6}$$

Then from Theorem 1.2 and Corollary 1.3 we obtain

Corollary 1.4. *Equation* (0.1) *is disconjugate on I if*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \le 0 \ \ \text{for} \ \ t \in I,$$

and is oscillatory on I if

$$p(t) \le -\frac{\lambda_2^4}{(b-a)^4} \ \ \text{for} \ \ t \in I.$$

Remark 1.4. In Theorem 4.1 of paper [5] (see also [2, Theorems 3.5 and 3.6], [1, Subsection 4.1]) following is proved: let λ_2 be the first positive root of the equation $\tanh \frac{\lambda}{\sqrt{2}} = \tan \frac{\lambda}{\sqrt{2}}$ ($\lambda_2 \approx 5.553$). Then the equation $u^{(4)} = -\lambda^4 u$ is disconjugate on $[0,1]$ if $0 \le \lambda < \lambda_2$.

1.3 Disconjugacy of equation (0.1) with not necessarily constant sign coefficient

On the basis of Theorems 1.1 and 1.2, we can get the non-improvable results which guarantee the diconjugacy of equation (0.1) on I, when p is not necessarily constant sign function.

Theorem 1.3. *Let $p_* \in D_-(I)$ and $p^* \in D_+(I)$. Then for an arbitrary function $p \in L(I;\mathbb{R})$, such that*

$$p_*(t) \preccurlyeq -[p(t)]_-, \quad [p(t)]_+ \preccurlyeq p^*(t) \ \ \text{for} \ \ t \in I, \tag{1.7}$$

equation (0.1) *is disconjugate on I.*

The theorem is optimal in the sense that inequalities (1.7) can not be replaced by the condition $p_* \le p \le p^*$.

Corollary 1.5. *Let the functions $p_1 \in L(I;\mathbb{R}_0^-)$, $p_2 \in L(I;\mathbb{R}_0^+)$, be such that the equations*

$$u^{(4)}(t) = p_1(t)u(t), \quad u^{(4)}(t) = p_2(t)u(t) \tag{1.8}$$

are disconjugate on I, and

$$p_1(t) \le p(t) \le p_2(t) \ \ \text{for} \ \ t \in I. \tag{1.9}$$

Then equation (0.1) *is disconjugate on I.*

Remark 1.5. We can see that in Kondrat'ev's comparison second theorem:

Theorem 1.4 ([4, Theorem 2])**.** *Let the continuous functions $p_1, p_2 : [a,b] \to \mathbb{R}$ be such that equations* (1.8) *are disconjugate on I, and $p_1 \le p \le p_2$. Then equation* (0.1) *is disconjugate too.*

The permissible coefficients p_1 and p_2 should not necessarily be constant sign functions, while in Theorem 1.3 for the permissible coefficients p_1 and p_2 equations (1.8) should not necessarily be disconjugate. For this reason, for example, if $p(t) = \lambda_1^4[\cos(2\pi t/n)]_+ - \lambda_2^4[\cos(2\pi t/n)]_-$, then from Theorem 1.3 it follows the disconjugacy of equation (0.1) on $[0,1]$ for all $n \in N$ (see Corollary 1.6), while this fact does not follows from Kondrat'ev's theorem.

From Theorem 1.3 with $p_* := -\frac{\lambda_2^4}{(b-a)^4}$ and $p^* := \frac{\lambda_1^4}{(b-a)^4}$ we obtain

Corollary 1.6. *Let* $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *be the first eigenvalues of problems* (1.3) *and* (1.6), *respectively, and the function* $p \in L(I;\mathbb{R})$ *admits the inequalities*

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \ \ \textit{for} \ \ t \in I.$$

Then equation (0.1) *is disconjugate on* I.

Remark 1.6. If we take into account that $\lambda_1^4 \approx 501$ and $\lambda_2^4 \approx 951$, then it is clear that Corollary 1.6 significantly improves W. Coppel's well-known condition

$$\max_{t\in[a,b]} |p(t)| \leq \frac{128}{(b-a)^4}\,,$$

proved in [3, Theorem 1, p. 86], which for $p \in C(I;\mathbb{R})$ guarantees the disconjugacy of equation (0.1) on I.

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The Ambrosetti–Prodi Problem for First Order Periodic ODEs with Sign-Indefinite Nonlinearities

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The Ambrosetti–Prodi problem for an equation of the form

$$F(x)=s \tag{1}$$

consists of determining how varying the parameter s affects the number of solutions x. Usually, an Ambrosetti–Prodi type result yields the existence of a number s_0 such that (1) has zero, at least one or at least two solutions according to $s<s_0$, $s=s_0$ or $s>s_0$. This terminology has become current after the founding work by A. Ambrosetti and G. Prodi [1] in 1972. Since then Ambrosetti–Prodi type results have been proved for several classes of boundary value problems: a thorough bibliography would include nearly two hundred titles.

In this contribution, based on the very recent paper [7], we analize the simplest case of the scalar periodic ODE

$$x'=f(t,x) \tag{2}$$

and the associated periodic Ambrosetti–Prodi problem

$$x'=f(t,x)-s. \tag{3}$$

Throughout we assume that $s\in\mathbb{R}$ is a parameter and

(h_1) *$f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is T-periodic with respect to the first variable and satisfies the L^1-Carathéodory conditions.*

Hereafter, by a T-periodic solution of (2) or (3) it is meant a T-periodic function $x:\mathbb{R}\to\mathbb{R}$ which is locally absolutely continuous and satisfies the equation for a.e. $t\in\mathbb{R}$.

Under the coercivity condition

$$f(t,x)\to+\infty, \ \text{ as } \ |x|\to+\infty \ \text{ uniformly a.e. in } t, \tag{4}$$

the periodic Ambrosetti–Prodi problem for (3) has been investigated by several authors, since the early eighties until very recent years: we refer to the bibliographies in [5, 6, 8] for a rather complete list of references. Thanks to its simplicity, (3) is in fact a quite good sample problem: manifold techniques can be effectively tested on it and the obtained results can suggest possible extensions to more general and complicated contexts.

In the case where f is a Bernoulli-type nonlinearity, i.e.,

(h_2) *there exist $a,b\in L^1(0,T)$ and $p>0$ such that $f(t,x)=a(t)|x|^p+b(t)$ for a.e. $t\in[0,T]$ and all $x\in\mathbb{R}$,* the coercivity assumption (4) amounts to requiring that

$$\operatorname*{ess\,inf}_{[0,T]} a > 0.$$

However, when modeling, for instance, population dynamics, it is interesting to include cases where the function a vanishes on sets of positive measure or changes sign, in order to describe the occurrence of seasonal periods which inhibit or adversely affect the growth rate of the population under consideration. A real outbreak of papers devoted to the study of nonlinear problems which are indefinite in sign dates back to the eighties of the last century both in the PDEs and the ODEs settings, together with a parallel renewed interest towards ecological models (see, e.g., the monograph [2]).

First relevant progresses in relaxing the uniform coercivity assumption (4) were achieved in the recent papers [3, 8, 9]; precisely, the following result for equation (3) was obtained in [8].

Theorem 1 ([8, Theorem 3.3])**.** *Assume* (h_1),

(h_3) *there exist* $a, b \in L^1(0,T)$ *such that* $f(t,x) \geq a(t)|x| + b(t)$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}$,

(h_4) *there exists* $\overline{x} \in \mathbb{R}$ *such that* $\operatorname*{ess\,sup}_{t\in[0,T]} f(t,\overline{x}) < +\infty$,

(h_5) *for every* $K_1, K_2, \sigma \in \,]0,+\infty[$, *there exists* $d > 0$ *such that, for every* $x \in C^0([0,T])$ *with* $x(0) = x(T)$, *if*

$$\max_{[0,T]} |x| \leq K_1 \min_{[0,T]} |x| + K_2 \tag{5}$$

and either $\min_{[0,T]} x \geq d$ *or* $\max_{[0,T]} x \leq -d$, *then* $\int\limits_0^T f(t,x)\,dt > \sigma$.

Then, there exists $s_0 \in \mathbb{R}$ *such that equation* (3) *has zero, at least one or at least two* T*-periodic solutions according to* $s < s_0$, $s = s_0$ *or* $s > s_0$.

It is easy to check (see, e.g., [8, Corollary 4.1]) that (h_5) holds whenever the function a which appears in (h_3) satisfies both

(h_6) $\quad a(t) \geq 0$ *for a.e.* $t \in [0,T]$

and

(h_7) $\quad \int\limits_0^T a(t)\,dt > 0.$

Accordingly, condition (h_5) permits to consider nonlinearities which are just locally coercive, although bounded from below by a L^1-function.

In [7] we pushed further into the direction of relaxing the coercivity assumption on f, by showing that the non-negativity condition (h_6) can be dropped at all, while still achieving all the conclusions of Theorem 1. Namely, we proved the following result.

Theorem 2. *Assume* (h_1), (h_4),

(h_8) *there exist* $a, b \in L^1(0,T)$ *and* $p \in \,]0,1]$ *with* $f(t,x) \geq a(t)|x|^p + b(t)$ *for a.e.* $t \in [0,T]$ *and all* $x \in \mathbb{R}$,

and (h_7). *Then, there exists* $s_0 \in \mathbb{R}$ *such that equation* (3) *has zero, at least one or at least two* T*-periodic solutions according to* $s < s_0$, $s = s_0$ *or* $s > s_0$.

Assumptions (h_8) and (h_7) basically require f being coercive on the average and allow that

$$\text{both} \quad \lim_{|x|\to+\infty} f(t,x) = +\infty \quad \text{and} \quad \lim_{|x|\to+\infty} f(t,x) = -\infty \quad \text{on sets of positive measure.}$$

It is worth stressing on the other hand that condition (h_5) prevents f from exhibiting this behavior, at least if f has the Bernoulli-type structure (h_2), as expressed by the following statement.

Proposition 3. *Assume (h_2). Then, condition (h_5) is equivalent to conditions (h_6) and (h_7).*

The proof of Theorem 2 is based on the direct construction of lower and upper solutions. Thus, from the results in [5], it is possible to infer various information on the qualitative properties of the obtained solutions. Indeed, for each $s > s_0$, equation (3) has at least one T-periodic solution which is weakly asymptotically stable from below, at least one T-periodic solution which is weakly asymptotically stable from above and at least one weakly stable T-periodic solution (all these solutions may possibly coincide), as well as, in addition, at least one unstable T-periodic solution, while for $s = s_0$ it has at least one unstable solution.

A question that may arise looking at Theorem (2) is whether or not one can assume $p > 1$ in condition (h_8). The answer is in general negative as shown by the following statement obtained in [7].

Proposition 4. *Assume (h_1) and*

(h_{10}) *there exist $p > 1$, $I = [t_1, t_2] \subseteq [0, T]$ and $\delta > 0$ such that $f(t,x) \le -\delta|x|^p$ for a.e. $t \in I$ and all $x \in \mathbb{R}$.*

Then, there exists $\sigma \in \mathbb{R}$ such that, for all $s \ge \sigma$, equation (3) has no T-periodic solutions.

In spite of the negative result of Proposition 4, we proved in [7] a positive result provided that $f(\cdot, 0) = 0$ and s is sufficiently small.

Proposition 5. *Assume (h_1),*

(h_{11}) *$f(\cdot,0) = 0$ and there exist $a \in L^1(0,T)$ and $p > 1$ such that $f(t,x) \ge a(t)|x|^p$ for a.e. $t \in [0,T]$ and all $x \in \mathbb{R}$,*

and (h_7). Then, there exists $\sigma > 0$ such that, for all $s \in \,]0, \sigma[\,$, problem (3) has at least one positive T-periodic solution and at least one negative T-periodic solution.

Open problem. It remains open the question of knowing if conclusions similar to the above can be proven for boundary value problems associated with second order ODEs or PDEs: a preliminary step in this direction is given by the perturbative result established in [3, Proposition 5.1].

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Qualitative Behavior of Solutions of Impulsive Weakly Nonlinear Hyperbolic Equation

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1 Introduction

Beginning from the pioneering works of A. M. Samoilenko the theory of differential equations with impulses [1,6,9] provides a mathematical tools for describing the behavior of many evolutionary processes with instant changes. The important subclass of the systems with discontinuous trajectories are impulsive (discontinuous) dynamical systems, whose trajectories have jumps after intersection with a given surface M in the phase space [5]. In infinite-dimensional phase spaces the theory of global attractors is a basis for qualitative analysis of solutions [10]. The lack of continuous dependence on initial data in impulsive dynamical systems do not allow us to apply directly the methods of the theory of global attractors. Nevertheless, one can use more general concept of uniform attractor [8] in this case. By definition, uniform attractor is a compact uniformly attracting (w.r.t. bounded initial data) set which is minimal among all such sets. Existence of such a set in impulsive infinite dimensional case firstly was proved in [3] for weakly nonlinear parabolic equation. It turned out that in the case of infinitely many impulsive points along trajectories the uniform attractor Θ has non-empty intersection with impulsive set M. As a consequence, it is neither invariant no stable set w.r.t. the impulsive semi-flow. In the paper [4] a new stability concept was introduced basing on the properties of the set $\Theta\setminus M$. In the present paper we investigate stability of uniform attractor for the weakly nonlinear second order evolutionary problem with impulses.

2 Uniform attractors of impulsive semi-flows

Impulsive dynamical system on normed space E consists of continuous semigroup $V: R_+\times E\to E$, impulsive set $M\subset E$ and impulsive map $I: M\to E$. The phase point moves along trajectories of V until the moment τ when the phase point $x(t)$ reaches the set M. At that moment the point instantaneously moves into a new position $Ix(\tau)$.

We need the following assumptions [5]:

$$\begin{gathered} M \text{ is closed}, \;\; M\cap IM=\varnothing, \\ \forall\, x\in M, \;\; \exists\,\tau=\tau(x)>0, \;\; \forall\, t\in(0,\tau) \;\; V(t,x)\notin M, \\ \text{every impulsive trajectory is defined on } [0,+\infty). \end{gathered} \tag{2.1}$$

Let us introduce notations:

$$\forall\, x\in M \;\; Ix=x^+, \;\; \forall\, x\in E \quad M^+(x)=\Big(\bigcup_{t>0}V(t,x)\Big)\cap M.$$

If $M^+(x) \neq \varnothing$, then from the continuity of V we deduce that there exists a moment $s > 0$ such that
$$\forall\, t \in (0, s) \;\; V(t, x) \notin M, \;\; V(s, x) \in M.$$

Then the impulsive semiflow G is described by the following construction:

if for $x \in E$ and for every $t > 0$ $V(t, x) \notin M$, then
$$G(t, x) = V(t, x).$$

Otherwise,
$$G(t,x) = \begin{cases} V(t - t_n, x_n^+), & t \in [t_n, t_{n+1}), \\ x_{n+1}^+, & t = t_{n+1}, \end{cases} \tag{2.2}$$

where $t_0 = 0$, $t_{n+1} = \sum\limits_{k=0}^{n} s_k$, $x_{n+1}^+ = IV(s_n, x_n^+)$, $x_0^+ = x$, s_n is a moment of impulsive perturbation, which is characterized by inclusion $V(s_n, x_n^+) \in M$.

Formula (2.2) defines (not necessary continuous) semigroup $G : R_+ \times E \to E$, which is called impulsive semiflow.

We will use the following notations:
$$\begin{gathered} b(E) \;\; \text{is a set of all bounded subsets of } E; \\ \operatorname{dist}(A, B) = \sup_{x \in A} \inf_{y \in B} \|x - y\|_E; \\ O_\delta(A) = \big\{x \in E : \;\; \operatorname{dist}(x, A) < \delta\big\}. \end{gathered}$$

Definition 2.1 ([3]). A compact set $\Theta \subset E$ is called a uniform attractor of the impulsive semiflow G if

(1) Θ is uniformly attracting set, i.e.
$$\forall\, B \in b(E) \;\; \operatorname{dist}(G(t, B), \Theta) \to 0, \;\; t \to \infty;$$

(2) Θ is the minimal among all sets satisfying (1).

Theorem 2.1 ([3]). *Assume that the impulsive semiflow G is dissipative, i.e.,*
$$\exists\, B_0 \in b(E), \;\; \forall\, B \in b(E), \;\; \exists\, T = T(B), \;\; \forall\, t \geq T \;\; G(t, B) \subset B_0. \tag{2.3}$$

Then G has uniform attractor Θ if and only if G is asymptotically compact, i.e.,
$$\forall\, \{t_n \nearrow \infty\} \;\; \forall\, \{x_n\} \in b(E), \;\; \forall\, \{t_n \nearrow \infty\}, \;\; \forall\, \{x_n\} \in b(E) \;\; \{G(t_n, x_n)\} \;\; \textit{is precompact in } E. \tag{2.4}$$

Moreover, the following equality takes place
$$\Theta = \omega(B_0) := \bigcap_{\tau > 0} \overline{\bigcup_{t \geq \tau} G(t, B_0)}. \tag{2.5}$$

Definition 2.2 ([2]). A set $A \subset E$ is called stable with respect to semiflow G if
$$A = D^+(A) := \bigcup_{x \in A} \big\{y : \;\; y = \lim G(t_n, x_n), \; x_n \to x, \; t_n \geq 0\big\}. \tag{2.6}$$

Remark. As $A \subset D^+(A)$, so (2.6) is equivalent to $D^+(A) \subset A$.

It is known that for continuous semiflows a uniform attractor is invariant and stable in the sense (2.6). Our main goal is to prove that for impulsive semiflow generated by impulsive perturbed wave equation the uniform attractor Θ satisfies the property
$$D^+(\Theta \setminus M) \subset \overline{\Theta \setminus M}. \tag{2.7}$$

3 Impulsive wave equation

We consider the triplet of Hilbert spaces $V \subset H \subset V^*$ with compact and dense embedding. Let $\|\cdot\|$ and $(\cdot\,,\,\cdot)$ be the norm and scalar product in H, $A : V \to V^*$ be a linear continuous self-adjoint coercive operator. The function $\langle Au, u\rangle^{\frac{1}{2}}$ defines a norm in the space V, which is denoted by $\|u\|_V$.

We consider the following evolutionary problem ($\beta > 0$):

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} + 2\beta\,\dfrac{\partial y}{\partial t} + Ay = \varepsilon F(y), \\ y\big|_{t=0} = y_0 \in V, \\ y_t\big|_{t=0} = y_1 \in H, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ is a small parameter, $F : H \mapsto H$ is a given Lipschitz continuous map. It is known [10] that in the phase space $E = V \times H$ this problem generates continuous semigroup $V : R_+ \times E \to E$, where

$$\text{for } \ z_0 = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \in E, \ \ V(t, z_0) = z(t) = \begin{pmatrix} y(t) \\ y_t(t) \end{pmatrix}.$$

The norm in E is given by the equality:

$$\text{for } \ z = \begin{pmatrix} y \\ w \end{pmatrix} \in E, \ \ \|z\|_E = \|y\|_V + \|w\|.$$

Qualitative behavior of linear impulsive wave equation firstly was considered in [7]. It was shown that it is natural to consider an impulsive set as a level set of some seminorm l_p, where

$$\forall\, z \in E \ \ l_p(z) \to \|z\|_E, \ \ p \to \infty.$$

Let $\{\lambda_i\}, \{\psi_i\}$ be solutions of spectral problem:

$$\forall\, i \geq 1 \ \ A\psi_i = \lambda_i \psi_i, \ \ 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \ \ \lambda_i \to \infty, \ \ i \to \infty.$$

For $p \geq 1$ let us consider $l_p : E \to R$:

$$\text{for } \ z = \begin{pmatrix} y \\ w \end{pmatrix} \in E, \ \ l_p(z) = \Big(\sum_{i=1}^{p} \big\{ \lambda_i (y, \psi_i)^2 + (w, \psi_i)^2 \big\} \Big)^{\frac{1}{2}}.$$

For fixed $p \geq 1$, $a > 0$, $\mu > 0$ let us put

$$\begin{aligned} M &= \big\{ z \in E : \ \ l_p(z) = a \big\}, \\ M' &= \big\{ z \in E : \ \ l_p(z) = a(1+\mu) \big\}, \end{aligned} \tag{3.2}$$

$I : M \to M^{'}$ such that

$$\text{for } \ z = \sum_{i=1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i \in M,$$

$$I(z) \in \bigg\{ \sum_{i=1}^{p} \begin{pmatrix} c_i' \\ d_i' \end{pmatrix} \psi_i + \sum_{i=p+1}^{\infty} \begin{pmatrix} c_i \\ d_i \end{pmatrix} \psi_i : \ \ \sum_{i=1}^{p} \big\{ \lambda_i (c_i')^2 + (d_i')^2 \big\} = a^2 (1+\mu)^2 \bigg\}. \tag{3.3}$$

The main result is the following theorem.

Theorem 3.1. *For every impulsive map $I : M \mapsto M'$ satisfying* (3.3) *and for sufficiently small $\varepsilon > 0$ the impulsive problem* (3.1)–(3.3) *generates impulsive semiflow $G : R_+ \times E \to E$, which has uniform attractor Θ and* (2.7) *takes place.*

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Baire and Local Baire Classes of Functionals on the Space of Linear Differential Systems

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For a given $n \in \mathcal{N}$ let us consider the set $\mathcal{M}^n$ of linear systems

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \equiv [0, +\infty), \tag{1}$$

with continuous and bounded matrix-valued functions $A : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$, which we identify with the corresponding linear systems.

In the contemporary theory of Lyapunov exponents the set $\mathcal{M}^n$ is usually equipped with the *uniform* and *compact-open* topologies defined respectively by the metrics

$$\rho_U(A, B) = \sup_{t \in \mathbb{R}_+} \|A(t) - B(t)\| \ \text{ and } \ \rho_C(A, B) = \sup_{t \in \mathbb{R}_+} \min\big\{\|A(t) - B(t)\|, 2^{-t}\big\},$$

with $\|\cdot\|$ being a matrix norm (e.g., the spectral one). The resulting topological spaces will be denoted by $\mathcal{M}^n_U$ and $\mathcal{M}^n_C$. V. M. Millionshchikov [5, 6] proposed using the Baire classification of discontinuous functions [2] to describe the dependence of various characteristics of asymptotic behavior of solutions to linear differential systems on their coefficients.

To recall what the Baire classification is, it is convenient to introduce the following notation. Let M be a metric space and F a collection of functions $f : M \to \mathbb{R}$. For each $k \in \mathcal{N}_0 \equiv \mathcal{N} \sqcup \{0\}$ we define the function class $[F]_k$ by induction as follows:

1. The class $[F]_0$ coincides with the class F.
2. The class $[F]_k$ consists of functions $f : M \to \mathbb{R}$ that can be represented in the form

$$f(x) = \lim_{j \to \infty} f_j(x), \quad x \in M,$$

   where functions $f_j : M \to \mathbb{R}$, $j \in \mathcal{N}$, belong to the class $[F]_{k-1}$.

Definition 1. Let M be a metric space. For each number $k \in \mathcal{N}$ we define the k-th Baire class $\mathfrak{F}_k(M)$ by the equality $\mathfrak{F}_k(M) = [C(M)]_k$, where $C(M)$ is the set of all continuous functions $f : M \to \mathbb{R}$. Besides, for each $k \in \mathcal{N}$ we define the exact k-th Baire class $\check{\mathfrak{F}}_k(M)$ by the equality $\check{\mathfrak{F}}_k(M) = \mathfrak{F}_k(M) \setminus \mathfrak{F}_{k-1}(M)$.

Put simply, Definition 1 tells that a function f belongs to the k-th Baire class if there exists a k-indexed sequence of continuous functions such that f can be obtained by taking pointwise limits of this sequence for k times (as each of its indices successively tends to infinity). Thus, when considering functionals on the space of linear differential systems, it is natural from the viewpoint of applications to require that the values of the prelimit functionals can be calculated from information about the system on finite intervals of the time semiaxis. This argument leads to the following [7]

Definition 2. We say that a functional $\varphi : \mathcal{M}^n \to \mathbb{R}$ has a compact support if there exists a $T > 0$ such that $\varphi(A) = \varphi(B)$ whenever $A, B \in \mathcal{M}^n$ coincide on the interval $[0, T]$. The set of all functionals with compact support will be denoted by $\mathfrak{C}^n$.

Definition 3. The k-th class of formulas $\mathfrak{C}^n_k$ is defined by

$$\mathfrak{C}^n_k = \big[\mathfrak{F}_0(\mathcal{M}^n_C) \cap \mathfrak{C}^n\big]_k, \ \ k \in \mathcal{N}_0.$$

It was established in the paper [5] that the classes $\mathfrak{C}^n_k$ and $\mathfrak{F}_k(\mathcal{M}^n_C)$ coincide with each other for all $k \geq 1$. This fact emphasizes the importance of studying the compact-open topology on the space $\mathcal{M}^n$.

For a more detailed classification let us give the following

Definition 4. The (k, m)-th class of formulas $\mathfrak{C}^n_{k,m}$ is defined by

$$\mathfrak{C}^n_{k,m} = \big[\mathfrak{F}_k(\mathcal{M}^n_C) \cap \mathfrak{C}^n\big]_m, \ \ k, m \in \mathcal{N}_0.$$

It is of interest to consider inclusions between different classes of formulas just introduced. The inclusion criterion is provided by the following

Theorem 1. *Let $i, j, k, m \in \mathcal{N}_0$. The inclusion $\mathfrak{C}^n_{i,j} \subset \mathfrak{C}^n_{k,m}$ holds if and only if $j \leq k$ and $i + j \leq k + m$.*

The main subject of study in the theory of Lyapunov exponents is not arbitrary functionals on the space of linear systems but rather the invariants of action of various transformation groups, particularly, Lyapunov transformations [1, Ch. III, § 1]. Hence, we give the following

Definition 5. A continuously differentiable matrix-valued function $L : \mathbb{R}_+ \to R^{n\times n}$ is called a Lyapunov transformation if $L(t)$ is invertible for all $t \in \mathbb{R}_+$ and the condition

$$\sup_{t\geq 0} \big(\|L(t)\| + \|L^{-1}(t)\| + \|\dot{L}(t)\|\big) < \infty$$

is satisfied. Systems $A, B \in \mathcal{M}^n$ are said to be Lyapunov equivalent if there exists a Lyapunov transformation that reduces the system A into the system B. A functional $\varphi : \mathcal{M}^n \to \mathbb{R}$ is called a Lyapunov invariant if $\varphi(A) = \varphi(B)$ whenever A and B are Lyapunov equivalent. The set of all Lyapunov invariants will be denoted by $\mathfrak{L}^n$.

It was shown in the paper [4] that the set $[\mathfrak{C}^n]_1 \cap \mathfrak{L}^n$ contains only constants and that $[\mathfrak{C}^n]_k \not\supset \mathfrak{F}_{k+1}(\mathcal{M}^n_C) \cap \mathfrak{L}^n$. These statements are supplemented by the following

Theorem 2. *Let i, j, k, m be nonnegative integers and $j \geq 2$. In order for the inclusion $\mathfrak{C}^n_{i,j} \cap \mathfrak{L}^n \subset \mathfrak{C}^n_{k,m} \cap \mathfrak{L}^n$ to hold it is necessary that $m \geq j$ and sufficient that $m \geq j + 1$.*

Remark. In the case $j \geq 2$ it is unknown to the author whether the classes $\mathfrak{C}^n_{i,j} \cap \mathfrak{L}^n$ and $\mathfrak{C}^n_{k,j} \cap \mathfrak{L}^n$ coincide for different i and k.

Since continuity (or discontinuity) of a function varies from point to point in its domain, a local Baire classification of functions makes sense [8].

Definition 6. Let M be a metric space and $k \in \mathcal{N}_0$. We say that a function f belongs to the k-th Baire class at a point $x_0 \in M$ and write $f \in \mathfrak{F}_k(\mathcal{M}^n_C, x_0)$ if there exists a neighborhood U of the point x_0 such that the restriction of f to U belongs to the k-th Baire class. If, in addition, $f \notin \mathfrak{F}_{k-1}(\mathcal{M}^n_C, x_0)$, then we say that a function f belongs to the exact k-th Baire class at the point $x_0 \in M$ and write $f \in \check{\mathfrak{F}}_k(\mathcal{M}^n_C, x_0)$. If $f \in \check{\mathfrak{F}}_k(\mathcal{M}^n_C, x_0)$ for all $x_0 \in M$, the function f is said to be uniform of the k-th Baire class.

It is well known [8] that each Lyapunov exponent considered as a function on $\mathcal{M}_C^n$ is uniform (of the second Baire class).

It turns out that this property is shared by all Lyapunov invariants as shown by the following

Theorem 3. *Each Lyapunov invariant $\mathcal{M}_C^n \to \mathbb{R}$ that belongs to a certain exact Baire class is uniform of that class.*

By contrast, each Lyapunov exponent considered as a function on $\mathcal{M}_U^n$ belongs to the zeroth Baire class at some points and to the second Baire class at others. It is known that for the two lowest exponents there are no points at which either of them belongs to the exact first Baire class [3, 8]. For the rest exponents it is conjectured but not proved to date.

The question naturally arises which local Baire classes can a general Lyapunov invariant $\mathcal{M}_U^n \to \mathbb{R}$ belong to at different points?

Theorem 4. *For every $n \in \mathcal{N}$ there exist a Lyapunov invariant $\varphi : \mathcal{M}_U^n \to [0,1]$ and a set of points $\{A_i \in \mathcal{M}^n : \ i \in \mathcal{N}\}$ such that*

$$\varphi \in \bigcap_{i\in\mathcal{N}} \check{\mathfrak{F}}_i(\mathcal{M}_U^n, A_i).$$

Theorem 5. *For any integers $n \geq 1$ and $N \geq 2$ there exist a Lyapunov invariant $\varphi : \mathcal{M}^n \to [0,1]$ and a set of points $\{A_i \in \mathcal{M}^n : \ i = 1, \dots, N\}$ such that*

$$\varphi \in \bigcap_{i=1}^{N} \check{\mathfrak{F}}_i(\mathcal{M}_U^n, A_i) \cap \check{\mathfrak{F}}_N(\mathcal{M}_C^n).$$

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On Sturm-Type Theorems for Nonlinear Equations

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1 Introduction

In this paper we consider the nonlinear equation of higher ($n > 2$) order:

$$y^{(n)} + p(t, y, y', \dots, y^{n-1})|y|^k \operatorname{sgn} y = 0, \quad k \in (0, 1) \cup (1, \infty), \tag{1.1}$$

where, for some $m, M \in \mathbb{R}$, the inequalities $0 < m \le |p(t, \xi_1, \xi_2, \dots, \xi_n)| \le M < \infty$ hold, the function $p(t, \xi_1, \xi_2, \dots, \xi_n)$ is continuous and Lipschitz continuous in $(\xi_1, \xi_2, \dots, \xi_n)$.

We study some oscillatory properties of (1.1) and compare them, for different n, with oscillatory properties of the linear equation

$$y^{(n)} + p(t)y = 0. \tag{1.2}$$

We obtain Sturm-type theorems.

2 Sturm's and Kondratiev's theorems

We know the classical result on properties of zeros of solutions to a second order linear equation, called Sturm's theorem.

Theorem 2.1 (Sturm J. Ch. F.)**.** *Consider two linearly independent solutions to the equation*

$$y'' + Q(t)y = 0$$

with a continuous function $Q(t)$, and let one of the solutions have two consecutive zeros. Then there is exactly one zero of another solution between those consecutive zeros.

This result was generalized by V. A. Kondratiev for linear equations of higher order.

Theorem 2.2 (V. A. Kondratiev, 1959)**.** *Suppose that solution to equation*

$$y''' + p(t)y = 0,$$

where $p(t)$ is continuous and $p(t) > 0$ for every t (or $p(t) < 0$ for every t), has consecutive zeros x_1 and x_2. Then every other solution to the equation has no more than two zeros on $[x_1, x_2]$.

Theorem 2.3 (V. A. Kondratiev, 1959)**.** *Suppose that solution to equation*

$$y^{IV} + q(t)y = 0,$$

where $p(t)$ is continuous and $p(t) > 0$ for every t, has consecutive zeros x_1 and x_2. Then every other solution to the equation has no more than four zeros on $[x_1, x_2]$.

Theorem 2.4 (V. A. Kondratiev, 1959)**.** *Suppose that solution to equation*

$$y^{IV} + q(t)y = 0,$$

where $p(t)$ is continuous and $p(t) < 0$ for every t, has consecutive zeros x_1 and x_2. Then every other solution to the equation has no more than three zeros on $[x_1, x_2]$.

And for linear equations of fifth and higher order V. A. Kondratiev proved (1961) that for $n \geq 5$ and $p(t) \geq 0$ there exists a solution to

$$y^{(n)} + p(t)y = 0$$

with arbitrary number of zeros between two consecutive zeros of another solution (see [5, 6]).

3 Theorem for the nonlinear equation

We consider equation (1.1) to be a generalisation of equation (1.2). Equation (1.1), which is, in turn, a generalisation to Emden equation, was studied in the [1–4, 7–13], and from variety of results obtained, we derive a theorem that serves as an analogue of Sturm's and Kondratiev's theorems, but for the nonlinear equation.

Theorem 3.1. *Suppose that a solution to equation* (1.1)*, where $p(t, \xi_1, \xi_2, \dots, \xi_n) > 0$ (or n is odd and $p(t, \xi_1, \xi_2, \dots, \xi_n) < 0$), has consecutive zeros x_1 and x_2. If $k \in (0, 1) \cup (1, +\infty)$, then there exists a solution to* (1.1) *with arbitrary finite number of zeros on $[x_1, x_2]$. If $k \in (0, 1)$, then there exists a solution to* (1.1) *with countable set of zeros on $[x_1, x_2]$, and a solution with a set of zeros on $[x_1, x_2]$ with the cardinality of the continuum.*

For the nonlinear equation results are the same for every $n > 2$, unlike results for linear equations. Any number of zeros is possible, irregardless of n.

Remark 1. Equation with even order n and negative $p(t, \xi_1, \xi_2, \dots, \xi_n)$ require more research, although we already know that we can't expect same results. As [1, Chapter 7] shows, solutions even to $y^{(4)} - y^3 = 0$ and $y^{(4)} + y^3 = 0$ differ greatly in their behaviour.

Remark 2. When $k = 1$, equation (1.1), in general, is not linear, but as we get a linear equation as special case when $p(t, \xi_1, \xi_2, \dots, \xi_n)$ depends only on t, in general case we can expect properties, similar to Sturm's and Kondratiev's results, where possible number of zeros depends on n. Further research is required.

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On a Hybrid Non-Linear Jump Control Problem for Functional Differential Equations with State-Dependent Impulses

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We study a control problem for the hybrid non-linear functional differential boundary value problem which is, in some sense, inverse to problems investigated in [1,3]. The problem consists of two systems of differential equations

$$x'(t) = f\big(t, x(t), x(\beta_1(t)), x(\beta_2(t))\big), \quad t \in [a, \tau], \tag{1}$$

$$y'(t) = \phi(t, y(t)), \quad t \in [\tau, b] \tag{2}$$

on the intervals $[a, \tau]$ and $[\tau, b]$ respectively, where the switching time τ is such that

$$g(t, x(\tau-)) = 0, \tag{3}$$

of the non-linear two-point boundary condition

$$V(x(a), y(b)) = 0, \tag{4}$$

the jump condition at the time instant τ

$$y(\tau+) - x(\tau-) = \gamma, \tag{5}$$

and the additional two-point conditions

$$\begin{aligned} x_i(a) &= z_i, \quad i = 1, \dots, j, \\ y_k(b) &= \eta_k, \quad k = j+1, \dots, n, \end{aligned} \tag{6}$$

where $1 \le j \le n$ is fixed.

The values of the time instant τ and the size of the jump γ are not specified beforehand and remain unknown. Thus, problem (1)–(6) is to determine the unknown values of τ and γ so that the solutions of (1), (2) satisfy the non-linear boundary conditions (4), the jump condition (3), (5) and the two-point conditions (6).

We consider the single-jump case [1, 2], i.e., it is assumed that the switching time τ is unique, which means that there is only one intersection of the integral curve of system (1) with the barrier set

$$G = \big\{(t,x) \in [a,b] \times \mathbb{R}^n : \ g(t,x) = 0\big\}. \tag{7}$$

The impulse action in this problem is state-dependent since the time instant τ is determined by the intersection of the curve with the barrier. In contrast to [1, 2], the jump magnitude γ here is unknown and plays the role of a control parameter. By a solution of (1)–(6), we mean the triplet (u,τ,γ), where

$$u(t) = \begin{cases} x(t) & \text{if } \ t \in [a,\tau], \\ y(t) & \text{if } \ t \in (\tau,b] \end{cases} \tag{8}$$

is left-continuous. The pre-jump evolution of the solution is described by the functional differential equation (1) and its after-jump behaviour is characterized by the ordinary differential equation (2). Equation (1), generally speaking, may contain other types of functional terms, which can be treated in a similar way [4].

In (1), (2), $f : [a,b] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ and $\phi : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the Carathéodory conditions, $\beta_i : [a,\tau] \to [a,\tau]$, $i = 0,1$, are measurable, g is continuous.

We will use an approach similar to [1] and approximate a solution u of form (8) of problem (1)–(6) by suitable sequences of functions separately on the intervals before and after the time when the jump occurs. The jump time τ itself remains unknown and is treated as parameter the value of which is to be determined.

Let us fix an arbitrary point $\tau \in (a,b)$ and choose certain compact convex sets D_a, $D_{\tau-}$, $D_{\tau+}$, D_b, Γ and define the sets

$$\begin{aligned} D_{a,\tau} &:= \big\{(1-\theta)z + \theta\lambda : \ z \in D_a, \ \lambda \in D_{\tau-}, \ \theta \in [0,1]\big\}, \\ D_{\tau+,b} &:= \big\{(1-\theta)(\lambda+\gamma) + \theta\eta : \ \lambda \in D_{\tau-}, \ \gamma \in \Gamma, \ \eta \in D_b, \ \theta \in [0,1]\big\}. \end{aligned}$$

Our technique is based on the parametrization

$$\begin{gathered} z = \operatorname{col}\big(x_1(a), x_2(a), \dots, x_j(a), z_{j+1}, \dots, z_n\big), \\ \eta = \operatorname{col}\big(\eta_1, \eta_2, \dots, \eta_j, y_{j+1}(b), \dots, y_n(b)\big), \\ x(\tau-) = \operatorname{col}(\lambda_1, \lambda_2, \dots, \lambda_n) = \lambda, \end{gathered}$$

which, together with $\gamma = \operatorname{col}(\gamma_1, \gamma_2, \dots, \gamma_n)$ and $\tau \in (a,b)$, constitutes the set of parameters $(z,\lambda,\eta,\gamma,\tau)$ with

$$z \in D_a, \ \ \lambda \in D_{\tau-}, \ \ \eta \in D_b, \ \ \gamma \in \Gamma. \tag{9}$$

Let us fix non-negative vectors $\varrho^{(i)}$, $i = 0,1$, and put

$$\Omega_0(\varrho^{(0)}) = O_{\varrho^{(0)}}(D_{a,\tau}), \quad \Omega_1(\varrho^{(1)}) = O_{\varrho^{(1)}}(D_{\tau+,b}), \tag{10}$$

where $O_\varrho(D) := \bigcup\limits_{z\in D} O_\varrho(z)$ for $D \subset \mathbb{R}^n$ and $O_\varrho(z) := \{\xi \in \mathbb{R}^n : \ |\xi - z| \le \varrho\}$ stand for the corresponding componentwise neighbourhoods of a set and a vector.

Introduce the following two auxiliary parametrized two-point boundary value problems for the investigation of the pre-jump and after-jump equations

$$\begin{gathered} x'(t) = f\big(t, x(t), x(\beta_1(t)), x(\beta_2(t))\big), \ \ t \in [a,\tau]; \quad x(a) = z, \ x(\tau) = \lambda, \\ y'(t) = \phi(t, y(t)), \ \ t \in (\tau,b]; \quad y(\tau) = \lambda + \gamma, \ \ y(b) = \eta, \end{gathered} \tag{11}$$

where the time instant τ and the vectors z, λ, γ, and η are treated as free parameters. We suppose that the functions x and y in (11) should take values in the sets Ω_0 and Ω_1, respectively.

To study problems (11), by analogy to [1, 5], we introduce two sequences of functions $\{x_m(\,\cdot\,,\tau,z,\lambda):\ m\geq 0\}$ and $\{y_m(\,\cdot\,,\tau,\lambda,\gamma,\eta):\ m\geq 0\}$ by putting

$$\begin{aligned}x_0(t,\tau,z,\lambda)&=\Big(1-\frac{t-a}{\tau-a}\Big)z+\frac{t-a}{\tau-a}\,\lambda,\ \ t\in[a,\tau],\\ y_0(t,\tau,\lambda,\gamma,\eta)&=\Big(1-\frac{t-\tau}{b-\tau}\Big)(\lambda+\gamma)+\frac{t-\tau}{b-\tau}\,\eta,\ \ t\in(\tau,b]\end{aligned}$$

and

$$\begin{aligned}x_{m+1}(t,\tau,z,\lambda)&=x_0(t,\tau,z,\lambda)\\ &+\int\limits_a^t(Fx_m(\,\cdot\,,\tau,z,\lambda))(s)\,ds-\frac{t-a}{\tau-a}\int\limits_a^\tau(Fx_m(\,\cdot\,,\tau,z,\lambda))(s)\,ds,\ \ t\in[a,\tau],\ \ m\geq 0,\end{aligned}\tag{12}$$

and

$$\begin{aligned}y_{m+1}(t,\tau,\lambda,\gamma,\eta)&=y_0(t,\tau,\lambda,\gamma,\eta)\\ &+\int\limits_\tau^t(\Phi y_m(\,\cdot\,,\tau,z,\lambda))(s)\,ds-\frac{t-\tau}{b-\tau}\int\limits_t^b(\Phi y_m(\,\cdot\,,\tau,z,\lambda))(s)\,ds,\ \ t\in(\tau,b],\ \ m\geq 0,\end{aligned}\tag{13}$$

where

$$\begin{aligned}&(Fx)(t):=f\big(t,x(t),x(\beta_1(t)),x(\beta_2(t))\big),\ \ t\in[a,\tau],\\ &(\Phi y)(t):=\phi(t,y(t)),\ \ t\in(\tau,b].\end{aligned}$$

Assume that $\varrho^{(0)}$, $\varrho^{(1)}$ involved in (10) can be chosen so that

$$\varrho^{(0)}\geq\frac{\tau-a}{2}\,\delta_{[a,b],(\Omega_0(\varrho^{(0)}))^3}(f),\quad \varrho^{(1)}\geq\frac{b-\tau}{2}\delta_{[a,b],\Omega_1(\varrho^{(1)})}(\phi),\tag{14}$$

where $\delta_{[a,b],\Omega_1(\varrho^{(1)})}(\phi)$ is $1/2$ of the oscillation of ϕ over $[a,b]\times\Omega_1(\varrho^{(1)})$,

$$\delta_{[a,b],\Omega_1(\varrho^{(1)})}(\phi):=\frac12\Big(\operatorname*{ess\,sup}_{(t,x)\in[a,b]\times\Omega_1(\varrho^{(1)})}\phi(t,x)-\operatorname*{ess\,inf}_{(t,x)\in[a,b]\times\Omega_1(\varrho^{(1)})}\phi(t,x)\Big)$$

and the value $\delta_{[a,b],(\Omega_0(\varrho^{(0)}))^3}(f)$ is defined by analogy. Assume that f and ϕ satisfy the Lipschitz conditions with respect to the space variables

$$\begin{gathered}\big|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)\big|\leq K_0|x_1-x_2|+K_1|y_1-y_2|+K_2|z_1-z_2|,\\ |\phi(t,\xi_1)-\phi(t,\xi_2)|\leq L|\xi_1-\xi_2|,\end{gathered}\tag{15}$$

respectively, on the sets $[a,b]\times(\Omega_0(\varrho^{(0)}))^3$ and $[a,b]\times\Omega_1(\varrho^{(1)})$, and put

$$Q_0=\frac{\tau-a}{2}\,(K_0+K_1+K_2),\quad Q_1=\frac{b-\tau}{2}\,L.$$

Theorem 1. *Let there exist non-negative vectors $\varrho^{(0)}$, $\varrho^{(1)}$ with properties* (14) *such that* (15) *holds on the sets $[a,b]\times(\Omega_0(\varrho^{(0)}))^3$ and $[a,b]\times\Omega_1(\varrho^{(1)})$ and*

$$r(Q_0)<1,\quad r(Q_1)<1.$$

Then, for all fixed $\tau\in(a,b)$, $z\in D_a$, $\lambda\in D_{\tau-}$, $\gamma\in\Gamma$, $\eta\in D_b$:

1. *Functions* (12), (13) *are absolutely continuous on* $[a,\tau]$ *and* $(\tau,b]$ *for* $m \ge 0$.
2. $\{x_m(t,\tau,z,\lambda):\ t \in [a,\tau],\ m \ge 0\} \subset \Omega_0(\varrho^{(0)})$, $\{y_m(t,\tau,\lambda,\gamma,\eta):\ t \in [\tau,b],\ m \ge 0\} \subset \Omega_1(\varrho^{(1)})$.
3. $\{x_m(\,\cdot\,,\tau,z,\lambda):\ m \ge 0\}$ *and* $\{y_m(t,\tau,\lambda,\gamma,\eta):\ m \ge 0\}$ *converge to the limit functions* $x_\infty(\,\cdot\,,\tau,z,\lambda)$, $y_\infty(\,\cdot\,,\tau,\lambda,\gamma,\eta)$ *uniformly on* $[a,\tau]$ *and* $[\tau,b]$.
4. $x_\infty(\,\cdot\,,\tau,z,\lambda)$ *and* $y_\infty(\,\cdot\,,\tau,\lambda,\gamma,\eta)$ *are the solutions of the boundary value problems*

$$\begin{gathered} x'(t) = (Fx)(t) + \frac{1}{\tau - a}\bigg(\lambda - z - \int\limits_a^\tau (Fx)(s)\,ds\bigg), \\ x(a) = z, \quad x(\tau) = \lambda, \end{gathered} \tag{16}$$

$$\begin{gathered} y'(t) = \phi(t,y(t)) + \frac{1}{b - \tau}\bigg(\eta - \lambda - z - \int\limits_\tau^b \phi(s,y(s)))\,ds\bigg), \\ y(\tau) = \lambda + \gamma, \quad y(b) = \eta, \end{gathered} \tag{17}$$

and problems (16), (17) *have no other solutions with values in* $\Omega_0(\varrho^{(0)})$ *and* $\Omega_1(\varrho^{(1)})$.

5. *The following estimates hold:*

$$\begin{aligned} \big|x_\infty(t,\tau,z,\lambda) - x_m(t,\tau,z,\lambda)\big| &\le \frac{\tau - a}{2}\, Q_0^m (1_n - Q_0)^{-1} \delta_{[a,b],(\Omega_0(\varrho^{(0)}))^3}(f), \\ \big|y_\infty(t,\tau,\lambda,\gamma,\eta) - y_m(t,\tau,\lambda,\gamma,\eta)\big| &\le \frac{b - \tau}{2}\, Q_1^m (1_n - Q_1)^{-1} \delta_{[a,b],\Omega_1(\varrho^{(1)})}(\phi). \end{aligned}$$

Theorem 2. *If, under the above conditions, some values* $(\tau,z,\lambda,\gamma,\eta) \in (a,b) \times D_a \times D_{\tau-} \times \Gamma \times D_b$ *satisfy the system of* $3n+1$ *scalar determining equations*

$$\begin{gathered} \int\limits_a^\tau (Fx_\infty(\,\cdot\,,\tau,z,\lambda))(s)\,ds = \lambda - z, \quad g(\tau,\lambda) = 0, \\ \int\limits_\tau^b \phi\big(s, y_\infty(s,\tau,\lambda,\gamma,\eta)\big)\,ds = \eta - \lambda - \gamma, \quad V(z,\eta) = 0 \end{gathered} \tag{18}$$

and, in addition, $g(t, y_\infty(t,\tau,\lambda,\gamma,\eta)) \ne 0$ *for any* $t \in (\tau,b]$, *then the function*

$$u_\infty(t,\tau,z,\lambda,\gamma,\eta) := \begin{cases} x_\infty(t,\tau,z,\lambda) & \text{if } t \le \tau \\ y_\infty(t,\tau,\lambda\gamma,\eta) & \text{if } t > \tau \end{cases}$$

is a solution of the original problem (1)–(6) *with a single jump* γ *at the time* τ.

The solvability of (1)–(6) can be checked by studying the approximate determining system obtained from (18) after replacing ∞ by a certain m.

Let us apply the approach described above to the systems

$$\begin{aligned} & x_1'(t) = -\frac{4}{3}\,(x_2(t^2))^2, \\ & x_2'(t) = (1-t)^2\Big(1.8\,x_1\Big(\frac{t}{3}\Big) + \frac{2}{9}\,t\Big), \ \ t \in [0,\tau]; \\ & y_1'(t) = y_2(t) - \frac{1}{4}\,, \\ & y_2'(t) = \frac{1}{2}\,\Big(y_2(t) - \frac{1}{4}\Big)^2 - y_1(t) + 1, \ \ t \in [\tau,1], \end{aligned}$$

under the boundary conditions

$$x_1^2(a) + y_2(b) = 0.6, \quad x_2^2(a) + x_1(a)y_2(b) = 0.25, \tag{19}$$

$$x_1(a) = 0.43, \quad y_2(b) = 0.42 \tag{20}$$

and the jump condition (3), (5) with the barrier set (7) of the form $\{(t, x_1, x_2) :\ (x_1+\frac{1}{2})^2 - t^2 = 0.1\}$. Clearly, (19) is a particular case of (4) and (20) has form (6) with $j = 1$, $z_1 = 0.43$, $\eta_2 = 0.42$.

The admissible sets for the parameter values were choosen as

$$\begin{aligned}
D_a = D_{\tau-} = D_{a,\tau-} &= \big\{(x_1, x_2) :\ 0.27 \le x_1 \le 0.45,\ 0.2 \le x_2 \le 0.56\big\},\\
\Gamma &= \big\{(x_1, x_2) :\ 0.15 \le x_1 \le 0.25,\ -0.3 \le x_2 \le -0.2\big\},\\
D_{\tau+} = D_b = D_{\tau+,b} &= \big\{(x_1, x_2) :\ 0.495 \le x_1 \le 0.58,\ -0.2 \le x_2 \le -0.45\big\}.
\end{aligned}$$

Putting in (14) $\varrho^{(0)} = \operatorname{col}(0.8, 0.9)$, $\varrho^{(1)} = \operatorname{col}(0.4, 0.5)$, for the pre-jump and after-jump curves we obtain the domains of form (10)

$$\begin{aligned}
\Omega_0(\varrho^{(0)}) &= \big\{(x_1, x_2) :\ -0.53 \le x_1 \le 1.25,\ -0.7 \le x_2 \le 1.46\big\},\\
\Omega_1(\varrho^{(1)}) &= \big\{(x_1, x_2) :\ 0.095 \le x_1 \le 0.98,\ -0.3 \le x_2 \le 0.95\big\}.
\end{aligned}$$

Applying Maple 14, we carried out computations according to (12), (13) and, for several values of m, obtained from the corresponding approximate determining systems the numerical values for the parameters which are presented in the table below. Note that the third and the fourth columns contain the numerical values of the jump magnitude $\gamma = \operatorname{col}(\gamma_1, \gamma_2)$ and the seventh one shows approximate values of the jump time τ.

m	η_1	γ_1	γ_2	λ_1	λ_2	τ	z_2
0	0.54721759	0.2025730	−0.2247659	0.31711742	0.52806054	0.75344599	0.26343879
1	0.54721759	0.2238353	−0.2364152	0.29470392	0.52737158	0.72907772	0.26343879
2	0.54721759	0.2240919	−0.2382983	0.29444608	0.52911668	0.72879666	0.26343879
3	0.54721759	0.2240736	−0.2383145	0.29446539	0.52914210	0.72881771	0.26343879
4	0.54721759	0.2240731	−0.2383131	0.29446596	0.52914096	0.72881833	0.26343879
5	0.54721759	0.2240731	−0.2383131	0.29446598	0.52914093	0.72881835	0.26343879

The residual obtained by substituting the approximate solution of the fifth approximation into the pre-jump and after-jump equations is of order 10^{-7}.

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Some Properties of the Lyapunov, Perron, and Upper-Limit Stabilities of Differential Systems

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This report is a logical continuation of the previous report [15] by the same author.

For a given zero neighborhood G in the Euclidean space $\mathbb{R}^n$, we consider the system

$$\dot{x} = f(t,x), \ \ x \in G, \ \ f(t,0) = 0, \ \ t \in \mathbb{R}_+ \equiv [0,+\infty), \tag{1}$$

where the right-hand side satisfies the condition $f, f'_x \in C(\mathbb{R}_+ \times G)$ and the zero solution is allowed. Let $S_*(f)$ and $S_\delta(f)$ denote the set of all non-continuable solutions x of system (1), given by the initial conditions $|x(0)| \neq 0$ and $0 < |x(0)| < \delta$, respectively.

Definition 1. We say that system (1) (more precisely, its zero solution) possesses the following *upper-limit* property:

1) *stability* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that any solution $x \in S_\delta(f)$ satisfies the condition
$$\overline{\lim_{t\to+\infty}} |x(t)| < \varepsilon; \tag{2}$$

2) *partial stability* if for any $\varepsilon, \delta > 0$ at least one solution $x \in S_\delta(f)$ satisfies condition (2);

3) *asymptotic stability* if there exists $\delta > 0$ such that any solution $x \in S_\delta(f)$ satisfies the condition
$$\overline{\lim_{t\to+\infty}} |x(t)| = 0 \quad \Big(\iff \lim_{t\to+\infty} |x(t)| = 0 \Big); \tag{3}$$

4) *global stability* if all solutions $x \in S_*(f)$ satisfy condition (3);

5) *instability* if there is no upper-limit stability, that is there is an $\varepsilon > 0$ such that for any $\delta > 0$ at least one solution $x \in S_\delta(f)$ does not satisfy condition (2) (in particular, it is not defined on the whole semi-axis $\mathbb{R}_+$);

6) *complete instability* if there is no upper-limit partial stability, that is there are $\varepsilon, \delta > 0$ such that no solution $x \in S_\delta(f)$ satisfies condition (2);

7) *asymptotic instability* if there is no upper-limit asymptotic stability, that is for any $\delta > 0$ at least one solution $x \in S_\delta(f)$ does not satisfy condition (3);

8) *total instability* if for some $\varepsilon > 0$ no solution $x \in S_*(f)$ satisfies condition (2).

Definition 2. Analogously to Definition 1, we say that system (1) possesses the corresponding *Perron* property (*stability*, *partial stability*, *asymptotic stability*, *global stability*, *instability*, *complete instability*, *asymptotic instability*, *total instability*) if it has property 1)–8) (respectively) from Definition 1, in which the upper limits under conditions (2) and (3) are replaced by the lower limits everywhere. Similarly, system (1) possesses the corresponding *Lyapunov* property if it has respectively property 1)–8) from Definition 1, in which:

(a) the upper limit at $t \to +\infty$ in condition (2) is replaced everywhere by an exact upper bound over all $t \in \mathbb{R}_+$;

(b) the requirement of Lyapunov stability is added to asymptotic and global stabilities in properties 3 and 4, respectively;

(c) property (7) is replaced by negation of the resulting property (3), that is either for any $\delta > 0$ at least one solution $x \in S_\delta(f)$ does not satisfy condition (3) or there is no Lyapunov stability.

We will be especially interested in particular cases of n-dimensional system (1): *one-dimensional* $(n = 1)$ and *two-dimensional* $(n = 2)$ systems, *autonomous* system

$$\dot{x} = f(x), \ \ f(0) = 0, \ \ x \in G \subset \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \tag{4}$$

with right-hand side $f \in C^1(G)$, and *linear* system

$$\dot{x} = A(t)x, \ \ x \in G \equiv \mathbb{R}^n, \ \ t \in \mathbb{R}_+, \tag{5}$$

defined by its continuous operator function $A : \mathbb{R}_+ \to \operatorname{End} \mathbb{R}^n$.

According to the next two theorems, the Lyapunov complete and total instabilities are equivalent, but this statement does not carry over the Perron instabilities and the upper-limit ones.

Theorem 1. *If system* (1) *is Lyapunov completely unstable, then it is Lyapunov totally unstable too.*

Theorem 2. *There exists a two-dimensional system* (1)*, which is Perron and upper-limit completely unstable, but neither Perron nor upper-limit totally unstable; moreover, it has at least one solution $x \in S_*(f)$ satisfying condition* (3)*.*

It can be seen from the following two theorems that, in the linear case, the assertion of Theorem 1 extends also to the complete and total instability of both Perron and upper-limit types, as well as to asymptotic and global stability of any type at all.

Theorem 3. *If the linear system* (5) *is Lyapunov, or Perron, or upper-limit completely unstable, then it is respectively Lyapunov, or Perron, or upper-limit totally unstable too.*

Theorem 4. *If the linear system* (5) *is Lyapunov, or Perron, or upper-limit asymptotically stable, then it is respectively Lyapunov, or Perron, or upper-limit globally stable too.*

In the autonomous case, Theorem 1 can be significantly reinforced, which is what the following two theorems do.

Theorem 5. *If for the autonomous system* (4) *at least one of the following six properties is satisfied: the Perron, Lyapunov, or upper-limit complete or total instability, then the other five of them are also satisfied.*

Theorem 6. *If the autonomous system* (4) *is not, at least, Lyapunov, or Perron, or upper-limit totally unstable, then it is both Lyapunov, and Perron, and upper-limit partially stable.*

Each of the upper-limit properties occupies a logically intermediate position between its Lyapunov and Perron analogs.

According to the following two theorems, in the one-dimensional and in the linear cases, all the upper-limit properties are indistinguishable from the corresponding Lyapunov properties, and under the additional condition of autonomy, also from the Perron ones.

Theorem 7. *For any one-dimensional system* (1)*, each of its upper-limit properties is equivalent to the analogous Lyapunov property, and in the case of an autonomous one-dimensional system, it is equivalent to the Perron property too.*

Theorem 8. *For any linear system* (5)*, each of its upper-limit properties is equivalent to the analogous Lyapunov property, and in the case of autonomic linear system, it is equivalent to the Perron property too.*

Already in the linear case, the upper-limit properties, although they coincide with the Lyapunov ones, can be directly opposite to the Perron ones.

Theorem 9. *For each $n \in \mathbb{N}$, there exists a linear n-dimensional system* (1)*, which is globally Perron stable, but both Lyapunov and upper-limit totally unstable.*

If the system is not linear and not one-dimensional, then even in the autonomous case, the upper-limit properties can also sharply contrast with both the Lyapunov and Perron ones.

Theorem 10. *There exists a two-dimensional autonomous system* (4)*, which is Lyapunov unstable, but both Perron and upper-limit globally stable.*

Theorem 11. *There exists a two-dimensional autonomous system* (4)*, which is globally Perron stable, but both Lyapunov and upper-limit unstable.*

Theorems 10 and 11 cannot be strengthened by replacing instability in them with complete instability (and even more so with total instability, this would contradict Theorem 5). The next theorem serves as a certain modification of Theorem 11.

Theorem 12. *There exists a two-dimensional autonomous system* (4)*, which is Perron stable, but both Lyapunov and upper-limit unstable; moreover, all its fixed points fill some ray $C \subset G$ with origin at zero, and any of its solutions $x \in S_*(f)$ with initial values $x(0) \notin C$ satisfies the relations*

$$0 = \varliminf_{t\to+\infty} |x(t)| < \varlimsup_{t\to\infty} |x(t)| = +\infty.$$

The system from Theorem 10 is described in [2, p. 6.3], and its simplified version is in [3, § 18].

For more information on these issues, see the reports [1, 4–6, 9–12, 14, 17–19]. The proofs of the above Theorems 1–12 are mainly contained in the papers [7, 8, 13, 16, 20].

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On One Class of Solutions of Linear Matrix Equations

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Let

$$G(\varepsilon_0)=\big\{t,\varepsilon:\ t\in\mathbf{R},\ \varepsilon\in(\mathbf{0},\varepsilon_{\mathbf{0}}),\ \varepsilon_{\mathbf{0}}\in\mathbf{R}^+\big\}.$$

Definition 1. We say that a function $f(t,\varepsilon)$ belongs to the class $S(m;\varepsilon_0)$, $m\in\mathbf{N}\cup\{\mathbf{0}\}$, if:

1) $f:G(\varepsilon_0)\to\mathbf{C}$,
2) $f(t,\varepsilon)\in C^m(G(\varepsilon_0))$ at t,
3) $d^kf(t,\varepsilon)/dt^k=\varepsilon^kf_k(t,\varepsilon)$ $(0\le k\le m)$,

$$\|f\|_{S(m;\varepsilon_0)}\overset{def}{=}\sum_{k=0}^{m}\sup_{G(\varepsilon_0)}|f_k(t,\varepsilon)|<+\infty.$$

Definition 2. We say that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belongs to the class $F(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$, if

$$f(t,\varepsilon,\theta(t,\varepsilon))=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)),$$

and

1) $f_n(t,\varepsilon)\in S(m,\varepsilon_0)$ $(n\in\mathbf{Z})$,
2)
$$\|f\|_{F(m;\varepsilon_0;\theta)}\overset{def}{=}\sum_{n=-\infty}^{\infty}\|f_n\|_{S(m;\varepsilon_0)}<+\infty,$$
3) $\theta(t,\varepsilon)=\int\limits_0^t\varphi(\tau,\varepsilon)\,d\tau$, $\varphi\in\mathbf{R}^+$, $\varphi\in S(m,\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

Definition 3. We say that a matrix $A(t,\varepsilon)=(a_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$ belongs to the class $S_2(m;\varepsilon_0)$ $(m\in\mathbf{N}\cup\{0\})$, if $a_{jk}\in S(m;\varepsilon_0)$ $(j,k=\overline{1,N})$.

We define the norm

$$\|A(t,\varepsilon)\|_{S_2(m;\varepsilon_0)}=\max_{1\le j\le N}\sum_{k=1}^{N}\|a_{jk}(t,\varepsilon)\|_{S(m;\varepsilon_0)}.$$

Definition 4. We say that a matrix $B(t,\varepsilon,\theta)=(b_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,N}}$ belongs to the class $F_2(m;\varepsilon_0;\theta)$ $(m\in\mathbf{N}\cup\{0\})$, if $b_{jk}(t,\varepsilon,\theta)\in F(m;\varepsilon_0;\theta)$ $(j,k=\overline{1,N})$.

We define the norm

$$\|B(t,\varepsilon,\theta)\|_{F_2(m;\varepsilon_0;\theta)} = \max_{1\le j\le N}\sum_{k=1}^{N}\|b_{jk}(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}. \tag{1}$$

Consider the linear non-homogeneous matrix equation

$$\frac{dX}{dt} = A(t,\varepsilon)X - XB(t,\varepsilon) + F(t,\varepsilon,\theta), \tag{2}$$

$A(t,\varepsilon), B(t,\varepsilon) \in S_2(m;\varepsilon_0)$, $F(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$.

We study the existence of particular solutions of equation (2) in the class $F(m_1;\varepsilon_1;\theta)$ ($m_1 \le m$, $\varepsilon_1 \le \varepsilon_0$).

Lemma. *Let*

$$\frac{dx}{dt} = \lambda(t,\varepsilon)x + u(t,\varepsilon,\theta(t,\varepsilon)) \tag{3}$$

be a given scalar linear non-homogeneous first-order differential equation, where $\lambda(t,\varepsilon) \in S(m;\varepsilon)$, $\inf\limits_{G(\varepsilon_0)}|\operatorname{Re}\lambda(t,\varepsilon)| = \gamma > 0$, *and* $u(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$*. Then equation* (3) *has a unique particular solution* $x(t,\varepsilon,\theta) \in F(m;\varepsilon_0;\theta)$*. This solution is given by the formula*

$$x(t,\varepsilon,\theta(t,\varepsilon)) = \int_T^t u(\tau,\varepsilon,\theta(\tau,\varepsilon))\exp\bigg(\int_\tau^t \lambda(s,\varepsilon)\,ds\bigg)\,d\tau,$$

where

$$T = \begin{cases} -\infty & \text{if } \operatorname{Re}\lambda(t,\varepsilon) \le -\gamma < 0,\\ +\infty & \text{if } \operatorname{Re}\lambda(t,\varepsilon) \ge \gamma > 0.\end{cases}$$

Moreover, there exists $K_0 \in (0,+\infty)$ *such that*

$$\|x(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)} \le K_0\|u(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

Theorem 1. *Let equation* (2) *satisfy the next conditions:*

1) *there exist matrices* $L_1(t,\varepsilon), L_2(t,\varepsilon) \in S_2(m;\varepsilon_0)$ *such that*

   (a) $\det L_1(t,\varepsilon) \ge a_0 > 0$, $\det L_2(t,\varepsilon) \ge a_0 > 0$;

   (b) $L_1^{-1}(t,\varepsilon)A(t,\varepsilon)L_1(t,\varepsilon) = D_1(t,\varepsilon) = (d^1_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$,

   (c) $L_2(t,\varepsilon)B(t,\varepsilon)L_2^{-1}(t,\varepsilon) = D_2(t,\varepsilon) = (d^2_{jk}(t,\varepsilon))_{j,k=\overline{1,N}}$,

   where D_1, D_2 *are lower triangular matrices belonging to the class* $S_2(m;\varepsilon_0)$*;*

2) $\inf\limits_{G(\varepsilon_0)}|\operatorname{Re}(d^1_{jj}(t,\varepsilon) - d^2_{kk}(t,\varepsilon))| \ge b_0 > 0$ ($j,k = \overline{1,N}$).

Then there exists $\varepsilon_1 \in (0,\varepsilon_0)$ *such that for all* $\varepsilon \in (0,\varepsilon_1)$ *there exists unique particular solution* $X(t,\varepsilon,\theta) \in F_2(m-1;\varepsilon_1;\theta)$ *of the matrix equation* (2).

Proof. We make in equation (2) the substitution

$$X = L_1(t,\varepsilon)Y(t,\varepsilon)L_2(t,\varepsilon), \tag{4}$$

where Y – the new unknown matrix. We obtain

$$\frac{dY}{dt}=\Big(D_1(t,\varepsilon)-L_1^{-1}(t,\varepsilon)\,\frac{dL_1(t,\varepsilon)}{dt}\Big)Y \\ -Y\Big(D_2(t,\varepsilon)+\frac{dL_2(t,\varepsilon)}{dt}\,L_2^{-1}(t,\varepsilon)\Big)+L_1^{-1}(t,\varepsilon)F(t,\varepsilon,\theta)L_2^{-1}(t,\varepsilon). \tag{5}$$

We denote

$$L_1^{-1}(t,\varepsilon)\,\frac{dL_1(t,\varepsilon)}{dt}=\varepsilon H_1\,\frac{dL_1(t,\varepsilon)}{dt}\,,\quad \frac{dL_2(t,\varepsilon)}{dt}\,L_2^{-1}(t,\varepsilon)=\varepsilon H_2\,\frac{dL_2(t,\varepsilon)}{dt}\,L_2^{-1}(t,\varepsilon), \\ L_1^{-1}(t,\varepsilon)F(t,\varepsilon,\theta)L_2^{-1}(t,\varepsilon)=F_1(t,\varepsilon,\theta).$$

Then equation (5) may be written as

$$\frac{dY}{dt}=D_1(t,\varepsilon)Y-YD_2(t,\varepsilon)-\varepsilon H_1(t,\varepsilon)Y-\varepsilon YH_2(t,\varepsilon)+F_1(t,\varepsilon,\theta).$$

By virtue Lemma and condition 2) of the theorem, the equation

$$\frac{dY_0}{dt}=D_1(t,\varepsilon)Y_0-Y_0D_2(t,\varepsilon)+F_1(t,\varepsilon,\theta)$$

has a unique solution $Y_0(t,\varepsilon,\theta)$ of the class $F(m;\varepsilon_0;\theta)$, and there exists $K_1\in(0,+\infty)$ such that

$$\|Y_0(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}\le K_1\|F_1(t,\varepsilon,\theta)\|_{F(m;\varepsilon_0;\theta)}.$$

We construct the process of successive approximations, defining the initial approximation $Y_0(t,\varepsilon,\theta)$ and subsequent approximations defining as the solutions of the class $F(m-1;\varepsilon_0;\theta)$ of the equations

$$\frac{dY_k}{dt}=D_1(t,\varepsilon)Y_k-Y_kD_2(t,\varepsilon)-\varepsilon H_1(t,\varepsilon)Y_{k-1}-\varepsilon Y_{k-1}H_2(t,\varepsilon)+F_1(t,\varepsilon,\theta). \tag{6}$$

Using the ordinary technique of the contraction mapping principle it is easy to show that there exists $\varepsilon_1\in(0,\varepsilon_0)$ such that for all $\varepsilon\in(0,\varepsilon_1)$ process (6) convergence by the norm (1) to the solution of the class $F(m-1;\varepsilon_1;\theta)$ of equation (2).

Boundary Value Problems for Multi-Term Fractional Differential Equations with ϕ-Laplacian at Resonance

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1 Introduction

Let $T>0$ be given, $J=[0,T]$, $X=C(J)\times\mathbb{R}$ and $\|x\|=\max\{|x(t)|:\ t\in J\}$ be the norm in $C(J)$.

Let ϕ be an increasing and odd homeomorphism with $\phi(\mathbb{R})=\mathbb{R}$. The special case of ϕ is p-Laplacian $\phi_p(x)=|x|^{p-2}x$, $p>1$.

We discuss the fractional boundary value problem

$$ {}^c\!D^\alpha\phi\big({}^c\!D^\beta x(t)-a(t){}^c\!D^{\gamma_1}x(t)-b(t){}^c\!D^{\gamma_2}x(t)\big)=f(t,x(t)), \tag{1.1}$$

$$ x(0)=x(T),\quad {}^c\!D^\beta x(t)\big|_{t=0}=0, \tag{1.2}$$

where $\alpha\in(0,1]$, $0<\gamma_2<\gamma_1<\beta\le 1$, $a,b\in C(J)$, $f\in C(J\times\mathbb{R})$ and ${}^c\!D$ denotes the Caputo fractional derivative.

Definition 1.1. We say that $x:J\to\mathbb{R}$ is *a solution of equation* (1.1) if $x,{}^c\!D^\beta x\in C(J)$ and (1.1) holds for $t\in J$. A solution x of (1.1) satisfying the boundary condition (1.2) is called *a solution of problem* (1.1), (1.2).

We recall the definitions of the Riemann-Liouville fractional integral and the Caputo fractional derivative [2, 3].

The Riemann–Liouville fractional integral $I^\gamma x$ of order $\gamma>0$ of a function $x:J\to\mathbb{R}$ is defined as

$$ I^\gamma x(t)=\int_0^t\frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\,x(s)\,\mathrm{d}s,$$

where Γ is the Euler gamma function. I^0 is the identical operator.

The Caputo fractional derivative ${}^c\!D^\gamma x$ of order $\gamma>0$, $\gamma\notin\mathbb{N}$, of a function $x:J\to\mathbb{R}$ is given as

$$ {}^c\!D^\gamma x(t)=\frac{\mathrm{d}^n}{\mathrm{d}t^n}\int_0^t\frac{(t-s)^{n-\gamma-1}}{\Gamma(n-\gamma)}\Big(x(s)-\sum_{k=0}^{n-1}\frac{x^{(k)}(0)}{k!}\,s^k\Big)\,\mathrm{d}s,$$

where $n=[\gamma]+1$, $[\gamma]$ means the integral part of the fractional number γ. If $\gamma\in\mathbb{N}$, then ${}^c\!D^\gamma x(t)=x^{(\gamma)}(t)$. In particular,

$$ {}^c\!D^\gamma x(t)=\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t\frac{(t-s)^{-\gamma}}{\Gamma(1-\gamma)}\,(x(s)-x(0))\,\mathrm{d}s=\frac{\mathrm{d}}{\mathrm{d}t}\,I^{1-\gamma}(x(t)-x(0)),\ \ \gamma\in(0,1).$$

It is well known that $I^\gamma : C(J) \to C(J)$ for $\gamma \in (0,1)$; $I^\gamma I^\mu x(t) = I^{\gamma+\mu} x(t)$ for $x \in C(J)$ and $\gamma, \mu \in (0,\infty)$; ${}^c\!D^\gamma I^\gamma x(t) = x(t)$ for $x \in C(J)$ and $\gamma > 0$; if $x, {}^c\!D^\gamma x \in C(J)$ and $\gamma \in (0,1)$, then $I^\gamma {}^c\!D^\gamma x(t) = x(t) - x(0)$; if $0 < \beta < \alpha < 1$ and $x, {}^c\!D^\alpha x \in C(J)$, then ${}^c\!D^\beta x = I^{\alpha-\beta} {}^c\!D^\alpha x$.

Problem (1.1), (1.2) is at resonance, because every constant function x on J is a solution of problem ${}^c\!D^\alpha \phi({}^c\!D^\beta x - a(t){}^c\!D^{\gamma_1} x - b(t){}^c\!D^{\gamma_2} x) = 0$, (1.2).

The aim of this paper is to study the existence of solutions to problem (1.1), (1.2). To this end we first introduce an operator $\mathcal{Q} : C(J) \to C(J)$. Then, by $\mathcal{Q}$ an operator $\mathcal{L} : X \to X$ is defined and it is proved that if $(x, c) \in X$ is a fixed point of $\mathcal{L}$, then x is a solution of problem (1.1), (1.2). The existence of a fixed point of $\mathcal{L}$ is proved by the Schaefer fixed point theorem [1, 4].

We work with the following conditions for a, b and f in (1.1):

(H_1) $a(t) \geq 0$, $b(t) \geq 0$ for $t \in J$.

(H_2) There exist $D, H \in \mathbb{R}$, $D < 0 < H$, such that

$$f(t,x) < 0 \ \text{ for } \ t \in J, \ \ x \leq D,$$
$$f(t,x) > 0 \ \text{ for } \ t \in J, \ \ x \geq H.$$

(H_3) There exists a nondecreasing function $w : [0,\infty) \to (0,\infty)$ such that

$$\lim_{v\to\infty} \frac{1}{v}\, \phi^{-1}\Big(\frac{T^\alpha w(v)}{\Gamma(\alpha+1)}\Big) = 0$$

and

$$|f(t,x)| \leq w(|x|) \ \text{ for } \ (t,x) \in J \times \mathbb{R},$$

where ϕ^{-1} is the inverse function of ϕ.

2 Operator $\mathcal{Q}$ and its properties

The following result is the generalization of the Gronwall–Bellman lemma for singular kernels.

Lemma 2.1. *Let $0 < \zeta < \rho \leq 1$, $z \in C(J)$ be nonnegative and $c_1, c_2 \in [0,\infty)$. Suppose that $v \in C(J)$ is nonnegative and*

$$v(t) \leq z(t) + c_1 I^\zeta v(t) + c_2 I^\rho v(t), \ \ t \in J.$$

Then

$$v(t) \leq z(t) + d\Big(c_1 + \frac{c_2 \Gamma(\zeta) T^{\rho-\zeta}}{\Gamma(\rho)}\Big) I^\zeta z(t), \ \ t \in J,$$

where $d = d(\zeta, \rho)$ is a positive constant.

Let $\mathcal{F} : C(J) \to C(J)$ be the Nemytskii operator associated to f,

$$\mathcal{F}x(t) = f(t, x(t)).$$

For $x \in C(J)$, we discuss the auxiliary equation

$$u(t) = a(t) I^{\beta-\gamma_1} u(t) + b(t) I^{\beta-\gamma_2} u(t) + \phi^{-1} I^\alpha \mathcal{F}x(t) \tag{2.1}$$

with the unknown function u.

The following result is established by using Lemma 2.1 and the Schaefer fixed point theorem in $C(J)$.

Lemma 2.2. *Let $x \in C(J)$. Then equation* (2.1) *has a unique solution u in the set $C(J)$.*

Keeping in mind Lemma 2.2, for every $x \in C(J)$ there exists a unique solution $u \in C(J)$ of equation (2.1). We put $\mathcal{Q}x = u$ and have an operator $\mathcal{Q} : C(J) \to C(J)$ satisfying

$$\mathcal{Q}x(t) = a(t)I^{\beta-\gamma_1}\mathcal{Q}x(t) + b(t)I^{\beta-\gamma_2}\mathcal{Q}x(t) + \phi^{-1}I^{\alpha}\mathcal{F}x(t), \ \ x \in C(J). \tag{2.2}$$

The properties of $\mathcal{Q}$ are given in the following two lemmas.

Lemma 2.3. *Let* (H_1) *and* (H_2) *hold. Then*

$$\begin{aligned} x \in C(J), \ \ x(t) \le D \ \text{ on } J \implies \mathcal{Q}x(t) < 0 \ \text{ on } (0,T], \\ x \in C(J), \ \ x(t) \ge H \ \text{ on } J \implies \mathcal{Q}x(t) > 0 \ \text{ on } (0,T], \end{aligned}$$

Lemma 2.4. *Let* (H_3) *hold. Then $\mathcal{Q} : C(J) \to C(J)$ is continuous and*

$$\|\mathcal{Q}x\| \le E\phi^{-1}\Big(\frac{T^{\alpha}w(\|x\|)}{\Gamma(\alpha+1)}\Big), \ \ x \in C(J), \tag{2.3}$$

where

$$E = 1 + \frac{T^{\beta-\gamma_1}}{\Gamma(\beta-\gamma_1+1)}\,\Big(\|b\| + \frac{\|a\|\Gamma(\beta-\gamma_1)T^{\gamma_1-\gamma_2}}{\Gamma(\beta-\gamma_2)}\Big).$$

3 Operator $\mathcal{L}$ and its properties

Let an operator $\mathcal{L} : X \to X$ be defined by

$$\mathcal{L}(x,c) = \Big(c + I^{\beta}\mathcal{Q}x(t), c - I^{\beta}\mathcal{Q}x(t)\big|_{t=T}\Big).$$

The following two lemmas give the properties of $\mathcal{L}$.

Lemma 3.1. *If (x,c) is a fixed point of $\mathcal{L}$, then x is a solution to problem* (1.1), (1.2).

Proof. Let $(x,c) = \mathcal{L}(x,c)$ for some $(x,c) \in X$. Then

$$\begin{aligned} x(t) = c + I^{\beta}\mathcal{Q}x(t), \ \ t \in J, \\ I^{\beta}\mathcal{Q}x(t)\big|_{t=T} = 0, \end{aligned}$$

and therefore $x(0) = c$, $x(T) = c$ and ${}^c\!D^{\beta}x(t) = \mathcal{Q}x(t)$ for $t \in J$. Hence ${}^c\!D^{\beta}x \in C(J)$ and since $\mathcal{Q}x(t)|_{t=0} = 0$, we have ${}^c\!D^{\beta}x(t)|_{t=0} = 0$. Thus x satisfies the boundary condition (1.2) and

$${}^c\!D^{\gamma_1}x(t) = I^{\beta-\gamma_1}\,{}^c\!D^{\beta}x(t), \quad {}^c\!D^{\gamma_2}x(t) = I^{\beta-\gamma_2}\,{}^c\!D^{\beta}x(t), \ \ t \in J.$$

Combining these equalities with (2.2) and ${}^c\!D^{\beta}x(t) = \mathcal{Q}x(t)$ we obtain

$$\begin{aligned} {}^c\!D^{\beta}x(t) &= \mathcal{Q}x(t) = a(t)I^{\beta-\gamma_1}\mathcal{Q}x(t) + b(t)I^{\beta-\gamma_2}\mathcal{Q}x(t) + \phi^{-1}I^{\alpha}\mathcal{F}x(t) \\ &= a(t)I^{\beta-\gamma_1}\,{}^c\!D^{\beta}x(t) + b(t)I^{\beta-\gamma_2}\,{}^c\!D^{\beta}x(t) + \phi^{-1}I^{\alpha}\mathcal{F}x(t) \\ &= a(t)\,{}^c\!D^{\gamma_1}x(t) + b(t)\,{}^c\!D^{\gamma_2}x(t) + \phi^{-1}I^{\alpha}\mathcal{F}x(t), \ \ t \in J. \end{aligned}$$

In particular,

$${}^c\!D^{\beta}x(t) - a(t)\,{}^c\!D^{\gamma_1}x(t) - b(t)\,{}^c\!D^{\gamma_2}x(t) = \phi^{-1}I^{\alpha}\mathcal{F}x(t), \ \ t \in J.$$

Applying ϕ and then ${}^c\!D^{\alpha}$ on both its sides, it follows

$${}^c\!D^{\alpha}\phi\Big({}^c\!D^{\beta}x(t) - a(t)\,{}^c\!D^{\gamma_1}x(t) - b(t)\,{}^c\!D^{\gamma_2}x(t)\Big) = \mathcal{F}x(t), \ \ t \in J.$$

Hence x is a solution of equation (1.1). As a result, x is a solution to problem (1.1), (1.2). □

Lemma 3.2. *Let* (H_3) *hold. Then $\mathcal{L}$ is a completely continuous operator.*

4 Problem (1.1), (1.2)

Theorem 4.1. *Let* (H_1)–(H_3) *hold. Then problem* (1.1), (1.2) *has at least one solution.*

Proof. By Lemma 3.1, we need to prove that $\mathcal{L}$ has a fixed point. Since $\mathcal{L}$ is completely continuous by Lemma 3.2, the Schaefer fixed point theorem guarantees the existence of a fixed point of $\mathcal{L}$ if the set $\mathcal{U} = \{(x,c) \in X : \ (x,c) = \lambda\mathcal{L}(x,c) \text{ for some } \lambda \in (0,1)\}$ is bounded. We show that $\mathcal{U}$ is bounded. $\square$

Example 4.2. Let $\phi = \phi_p$, $p > 1$, $\mu \in (0, p-1)$, $r, m, k \in C(J)$ and $f(t,x) = k(t) + |x|^\mu \arctan x$. Then conditions (H_1) and (H_2) are satisfied for $a = |r|$, $b = |m|$, $H = \max\{\pi/4, \sqrt[\mu]{\|k\|}\}$ and $D = -H$. Since $\phi^{-1} = \phi_q$, $q = p/(p-1)$, condition (H_3) is fulfilled for $w(v) = \|k\| + \pi v^\mu/2$. Theorem 4.1 guarantees that the problem

$$
\begin{gathered}
{}^c\mathcal{D}^\alpha \phi_p\Big({}^c\mathcal{D}^\beta x - |r(t)|\,{}^c\mathcal{D}^{\gamma_1} x - |m(t)|\,{}^c\mathcal{D}^{\gamma_2} x\Big) = k(t) + |x|^\mu \arctan x, \\
x(0) = x(T), \quad {}^c\mathcal{D}^\beta x(t)\big|_{t=0} = 0,
\end{gathered}
$$

has a solution.

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Asymptotic Behavior of Stochastic Functional Differential Equations in Hilbert Spaces

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We study the long time behavior of nonlinear stochastic functional-differential equations in Hilbert spaces. In particular, we start with establishing the existence and uniqueness of mild solutions. We proceed with deriving a priory uniform in time bounds for the solutions in the appropriate Hilbert spaces. These bounds enable us to establish the existence of invariant measure based on the Krylov–Bogoliubov theorem on the tightness of the family of measures.

1 Introduction

In this work we study the asymptotic behaviour of solutions of stochastic functional-differential equations. In a bounded domain, the equation reads as

$$\begin{gathered} du=[Au+f(u_t)]\,dt+\sigma(u_t)\,dW(t)\ \text{ in } D,\ \ t>0; \\ u(t,x)=\phi(t,x),\ \ t\in[-h,0),\quad u(0,x)=\varphi_0(x)\ \text{ in } D; \\ u(t,x)=0,\ \ x\in\partial D,\ \ t\geq 0. \end{gathered} \tag{1.1}$$

The corresponding problem in the entire space has the form

$$\begin{gathered} du=[Au+f(u_t)]\,dt+\sigma(u_t)\,dW(t)\ \text{ in } \mathbb{R}^d,\ \ t>0; \\ u(t,x)=\phi(t,x),\ \ t\in[-h,0),\quad u(0,x)=\varphi_0(x)\ \text{ in } \mathbb{R}^d. \end{gathered} \tag{1.2}$$

Here A is an elliptic operator

$$A=A(x)=\sum_{i,j=1}^{d}a_{ij}(x)\,\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{d}b_i(x)\,\frac{\partial}{\partial x_i}+c(x), \tag{1.3}$$

the interval $[-h,0]$ is the interval of delay, and $u_t=u(t+\theta)$ with $\theta\in[-h,0)$.

Functional differential equations of types (1.1) and (1.2) are mathematical models of processes, the evolution of which depends on the previous states. The classical results for deterministic functional-differential equations in finite dimensional spaces can be found in [6] and the references therein. Stochastic functional differential equations in finite dimensions have been studied extensively as well. In particular, the existence of invariant measures for stochastic ordinary differential equations was established in [1] and [5].

The results on functional differential equations in infinite dimensions are significantly more sparse.

The main goal of the present work is to establish the existence and uniqueness of invariant measures for equations (1.1) and (1.2) based on the Krylov–Bogoliubov theorem on the tightness of the family of measures [7]. More precisely, we will use the compactness approach of Da Parto and Zabczyk [3], which involves the following key steps:

(i) Establishing the existence of a Markovian solution of (1.1) or (1.2) in a certain functional space, in which the corresponding transition semigroup is Feller;

(ii) Showing that the semigroup $S(t)$ generated by A is compact;

(iii) Showing that the corresponding equation with a suitable initial condition has a solution, which is bounded in probability.

This approach was used in establishing the existence of invariant measure for a large class of stochastic nonlinear partial differential equations without delay, e.g. [4] and references therein. For functional differential equations in finite dimensions, the approach above was used in [2]. In this work, the author established the existence of an invariant measure in $\mathbb{R}^d \times L^2(-h,0;\mathbb{R}^d)$. In contrast, for stochastic partial differential equations, the natural phase space for the mild solutions of (1.2) is $L^2_\rho(\mathbb{R}^d) \times L^2(-h,0,L^2_\rho(\mathbb{R}^d))$, where $L^2_\rho(\mathbb{R}^d)$ is a weighted space. The equations of type (1.1) and (1.2) were studied in the space $C(-1,0,L^2_\rho(\mathbb{R}^d))$, which is a significantly easier problem. In these spaces the authors studied the conditions for the existence and uniqueness of a solution, as well as their Markov's and Feller properties. However, in order to apply the compactness approach one needs to work in $L^2_\rho(\mathbb{R}^d) \times L^2(-h,0,L^2_\rho(\mathbb{R}^d))$, which is done in this work.

2 Formulation of the problem and the main result

Throughout the paper, the domain D is either a bounded domain with ∂D satisfying the Lyapunov condition, or $D = \mathbb{R}^d$. Denote

$$\rho(x) := \frac{1}{1+|x|^r}\,,$$

where $r > d$ if $D = \mathbb{R}^d$ and $r = 0$ (i.e. no weight) for bounded D. We introduce the following spaces:

$$B^\rho_0 := L^2_\rho(D), \quad B^\rho_1 := L^2(-h,0,L^2_\rho(D)), \quad B^\rho := B^\rho_0 \times B^\rho_1, \quad H := L^2(D).$$

The coefficients a_{ij} of the operator A defined in (1.3) are Holder continuous with the exponent $\beta \in (0,1)$, symmetric, bounded and satisfying the elipticity condition

$$\sum_{i,j=1}^d a_{i,j}\eta_i\eta_j \geq C_0|\eta|, \quad \eta \in \mathbb{R}^d$$

for some $C_0 > 0$. The coefficients b_i and c are also bounded and Holder continuous with some positive Holder exponent.

If D is bounded, we impose homogeneous Dirichlet boundary conditions on ∂D. In this case,

$$D(A) = H^2(D) \cap H^1_0(D).$$

If $D = \mathbb{R}^d$, then $D(A) = H^2(\mathbb{R}^d)$. Denote $G(t,x,y)$ to be the fundamental solution (or the Green's function in the case of bounded D) for $\frac{\partial}{\partial t} - A$. It is well known, that there are positive constants $C_1(T), C_2(T) > 0$ such that

$$0 \leq G(t,x,y) \leq C_1(T)t^{-d/2}e^{-C_2(T)\,\frac{|x-y|^2}{t}} \tag{2.1}$$

for $t \in [0,T]$ and $x,y \in D$. Note that in (2.1), C_1 and C_2 depend not only on T, but on the constants C_0, d, T, maximum values of the coefficients of A, and the Holder constants. If the operator is in the divergence form $Au = \operatorname{div}(a\nabla u)$, the estimates are of a different type, namely,

$$g_1(t,x-y) \leq G(t,x,y) \leq g_2(t,x-y),$$

where

$$g_i(t,x)=K(C_0,d)t^{-d/2}e^{-K(C_0,d)\,\frac{|x|^2}{t}},\ \ i=1,2,\ \ t\geq 0,\ \ x,y\in\mathbb{R}^d.$$

In this case, in contrast with (2.1), the constant $K(C_0,d)$ is independent of t.

Let $\sum\limits_{i=1}^{\infty}a_i<\infty$, and e_n be orthonormal basis in H, such that $e_n\in L^\infty(D)$ and $\sup\limits_n\|e_n\|_{L^\infty(D)}<\infty$. Introduce the operator $Q\in\mathcal{L}(H)$ such that Q is non-negative, $\operatorname{Tr}(Q)<\infty$, $Qe_n=a_ne_n$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space. Introduce

$$W(t):=\sum_{i=1}^{\infty}\sqrt{a_i}\beta_i(t)e_i(x),\ \ t\geq 0,$$

which is a Q-Wiener process on $t\geq 0$ with values in $L^2(Q)$. Here $\beta_i(t)$ are standard, one dimensional, independent Wiener processes. Also let $\{F_t,\ t\geq 0\}$ be a normal filtration satisfying

- $W(t)$ is $\mathcal{F}_t$-measurable;

- $W(t+h)-W(t)$ is independent of $\mathcal{F}_t$ $\forall\,h\geq 0,t\geq 0$.

Denote $U=Q^{\frac12}(H)$. It is well known $U\in L^\infty(D)$. Introduce the multiplication operator $\Phi:U\to B_0^\rho$ as follows: for fixed $\varphi\in B_0^\rho$, let $\Phi(\psi):=\varphi\psi$, $\psi\in U$. Since $\varphi\in B_0^\rho$ and $\varphi\in L^\infty(D)$, the operator is well defined and hence $\Phi\circ Q^{1/2}:L^2(D)\to B_0^\rho$ defines a Hilbert–Schmidt operator. The operator Φ is also a Hilbert–Schmidt operator satisfying

$$\|\Phi\circ Q^{1/2}\|_{\mathcal{L}_2}^2=\sum_{n=1}^{\infty}\|\Phi\circ Q^{1/2}e_n\|_{B_0^\rho}^2=\sum_{n=1}^{\infty}a_n\int\limits_D\varphi^2(x)e_n^2(x)\rho(x)\,dx\leq\operatorname{Tr}(Q)\sup_n\|e_n\|_\infty^2\|\varphi\|_\rho^2,$$

where $\operatorname{Tr}(Q)=\sum_{n=1}^\infty a_n=a$. Hence if $\Phi:\Omega\times[0,T]\to\mathcal{L}(U,B_0^\rho)$ is a predictable process satisfying

$$\mathbb{E}\int\limits_0^T\|\Phi\circ Q^{1/2}\|_{\mathcal{L}_2}^2\,ds<\infty,$$

following [3] we can define

$$\int\limits_0^t\Psi(s)\,dW(s)\in B_0^\rho$$

with the following expansion

$$\int\limits_0^t\Psi(s)\,dW(s)=\sum_{i=1}^{\infty}\sqrt{a_i}\int\limits_0^t\Phi(s,\,\cdot\,)e_i(\,\cdot\,)\,d\beta_i(s).$$

Furthermore,

$$\mathbb{E}\bigg\|\int\limits_0^t\Psi(s)\,dW(s)\bigg\|_\rho^2\leq a\sup_n\|e_n\|_\infty^2\int\limits_0^t\mathbb{E}\|\Psi(s,\,\cdot\,)\|_{B_0^\rho}^2\,ds.$$

Assumptions on nonlinearities

Assume f and σ satisfy the following conditions:

(i) The functionals f and σ map B_1^ρ to B_0^ρ;

(ii) There exists a constant $L > 0$ such that

$$\|f(\varphi_1) - f(\varphi_2)\|_{B_0^\rho} + \|\sigma(\varphi_1) - \sigma(\varphi_2)\|_{B_0^\rho} \leq L\|\varphi_1 - \varphi_2\|_{B_1^\rho}$$

for any $\varphi_1, \varphi_2 \in B_1^\rho$.

Definition. An $\mathcal{F}_t$ measurable random process $u(t, \cdot\,) \in B_0^\rho$ is a mild solution of (1.1) or (1.2) if

$$u(t, \cdot\,) = S(t)\varphi(0, \cdot\,) + \int\limits_0^t S(t-s)f(u_s)\,ds + \int\limits_0^t S(t-s)\sigma(u_s)\,dW(s), \tag{2.2}$$

where

$$u(0, \cdot\,) = \varphi(0, \cdot\,) \in B_0^\rho, \;\; u(t, \cdot\,) = \varphi(t, \cdot\,) \in B_1^\rho, \;\; t \in [-h, 0].$$

Theorem 1 (Existence and uniqueness)**.** *Suppose f and σ satisfy conditions* (i) *and* (ii)*, and $\varphi(t, \cdot\,)$ is an $\mathcal{F}_0$ measurable random process for $t \in [-h, 0]$, which is independent of W and such that*

$$\mathbb{E}\|\varphi(0, \cdot\,)\|^p_{B_0^\rho} < \infty \;\; \textit{and} \;\; \mathbb{E}\|\varphi(\,\cdot\,, \cdot\,)\|^p_{B_1^\rho} < \infty, \;\; p \geq 2.$$

Then there exists a unique mild solution of (1.1) (*or* (1.2)) *on* $[0, T]$*, and*

$$\mathbb{E}\|y(t)\|^p_{B^\rho} \leq K(T)\big(1 + \mathbb{E}\|y(0)\|^p_{B^\rho}\big), \;\; t \in [0, T].$$

Define $\overline{\rho}(x) = (1 + |x|^{\overline{r}})^{-1}$. The main result of the paper is the following theorem.

Theorem 2. *Let the assumptions of Theorem 1 hold. Assume equation* (2.2) *has a solution in $B^{\overline{\rho}}$ which is bounded in probability for $t \geq 0$ with $r > d + \overline{r}$. Then there exists an invariant measure μ on B^ρ, i.e.*

$$\int\limits_{B^\rho} P_t\varphi(x)\,d\mu(x) = \int\limits_{B^\rho} \varphi(x)\,d\mu \;\; \textit{for any} \;\; t \geq 0 \;\; \textit{and} \;\; \varphi \in C_b(B^\rho).$$

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Application of the Averaging Method to Solving Boundary Value Problems for Systems with Impulse Action at Non-Fixed Moments of Time

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The averaging method is applied to study the existence of solutions of boundary value problems for systems with impulse action at non-fixed moments of time. It is shown that if an averaged boundary value problem has a solution, then the original problem is solvable as well. Here the averaged system is a system of autonomous ordinary differential equations.

1 Introduction

The present paper deals with the following boundary value problem for a system of differential equations with impulse action at non-fixed moments of time:

$$\begin{gathered} \dot{x} = \varepsilon X(t,x), \ \ t \neq t_i(x), \\ \Delta x\big|_{t=t_i(x)} = \varepsilon I_i(x), \\ F\Big(x(0), x\Big(\frac{T}{\varepsilon}\Big)\Big) = 0. \end{gathered} \tag{1.1}$$

Here $\varepsilon > 0$ is a small parameter, $t_i(x) < t_{i+1}(x)$, $i = 1, 2, \dots$, are the moments of impulse, X and I_i are d-dimensional vector functions.

Assuming that there exist the limits

$$X_0(x) = \lim_{T\to\infty} \frac{1}{T} \int\limits_0^T X(t,x)\, dt \tag{1.2}$$

and

$$I_0(x) = \lim_{T\to\infty} \frac{1}{T} \sum_{0<t_i(x)<T} I_i(x), \tag{1.3}$$

we put problem (1.1) in correspondence with the averaged boundary value problem

$$\dot{y} = \varepsilon\big[X_0(y) + I_0(y)\big], \quad F\Big(y(0), y\Big(\frac{T}{\varepsilon}\Big)\Big) = 0, \tag{1.4}$$

or, on the slow time scale $\tau = \varepsilon t$,

$$\frac{dy}{d\tau} = X_0(y), \quad F(y(0), y(T)) = 0.$$

The main result of this paper is a proof of the following statement: if the averaged boundary value problem has a solution, then, for small values of parameter ε, the original boundary value problem (1.1) also has a solution, and there is a proximity between their solutions.

Boundary value problems for systems with impulse action have been considered by many authors. To our knowledge, these problems were first studied in [3] when investigating periodical solutions by using the Samoilenko numerical-analytic method. Boundary value problems for systems with non-fixed moments of impulse were studied in [1] for the case of linear boundary conditions, and in [2] for the nonlinear case.

In the theory of ordinary differential equations, the method of averaging was first applied to boundary value problems in [4]. This method made it possible to reduce a boundary value problem for a non-autonomous system to an analogous problem for an autonomous averaged system. In the present paper, we apply this idea to solving the boundary value problem (1.1).

2 Formulation of the problem and the main result

We consider problem (1.1) under the assumption that the following conditions are satisfied:

(1) The functions $X(t, x)$ and $I_i(x)$ are uniformly continuous in a domain $Q = \{t \geq 0,\ x \in D \subset \mathbb{R}^d\}$;

(2) The functions $X(t, x)$ and $I_i(x)$ are bounded by a constant $M > 0$ and, with respect to x, satisfy the Lipschitz condition with a constant $L > 0$;

(3) There exist uniform in $x \in D$ limits (1.2) and (1.3), as well as the limits

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \frac{\partial X(t, x)}{\partial x}\, dt = \frac{\partial X_0(x)}{\partial x}$$

and

$$\lim_{T\to\infty} \frac{1}{T} \sum_{0<t_i<T} \frac{\partial I_i(x)}{\partial x} = \frac{\partial I_0(x)}{\partial x}\,;$$

(4) There exists a constant $C > 0$ such that, for $t \geq 0$ and $x \in D$,

$$i(t, x) \leq Ct,$$

where $i(t, x)$ is the number of impulses on $(0, t)$, and

$$\inf_{x\in D} \tau_{k+1}(x) > \sup_{x\in D} \tau_k(x);$$

(5) The averaged problem (1.4) has a solution $y = y(\tau) = y(\varepsilon, \tau)$ that belongs to D together with some ρ-neighborhood, in which $F(x, y)$ has uniformly continuous partial derivatives $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$, and $\det \frac{\partial F_0(x_0)}{\partial x_0} \neq 0$, here $x_0 = y(0)$, $F_0(x_0) = F(x_0, y(T, x_0))$.

Theorem 1. *Let conditions* (1)–(5) *be satisfied. Then there exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$ *one can specify a function* $\xi = \xi(\varepsilon)$, $\varepsilon \to 0$, *such that the boundary value problem* (1.1) *has a unique solution* $x(t, \varepsilon)$ *in* $\xi(\varepsilon)$*-neighborhood of* $y(\varepsilon t)$, *i.e.,*

$$\big|x(t,\varepsilon) - y(\varepsilon t)\big| < \xi(\varepsilon), \ \ t \in \Big[0, \frac{T}{\varepsilon}\Big], \ \ \varepsilon \in (0, \varepsilon_0).$$

The outline of the proof is as follows.

I. We first consider the system with impulse effect at fixed moments t_i on $[0, \frac{T}{\varepsilon}]$:

$$\begin{aligned} \dot{x} &= \varepsilon X(t,x), \ \ t \neq t_i, \\ \Delta x\big|_{t=t_i} &= \varepsilon I_i(x(t_i)). \end{aligned} \tag{2.1}$$

For this system, we derive a variational equation linearized along its solution $x(t, x_0)$ $(x(0, x_0) = 0)$, i.e.,

$$\begin{aligned} \dot{z} &= \varepsilon\, \frac{\partial X(t, x(t,x_0))}{\partial x}\,, \ \ t \neq t_i, \\ \Delta z\big|_{t=t_i} &= \varepsilon\, \frac{\partial I_i(x(t_i, x_0))}{\partial x}\, z(t_i), \end{aligned} \tag{2.2}$$

where $z(t) = \frac{\partial X(t,x_0)}{\partial x_0}$. We then establish the proximity between the solution of (2.2) and the solution $\frac{\partial y(\varepsilon t, x_0)}{\partial x_0}$ of the variational equation for the averaged system (under respective initial conditions).

II. By using the implicit function theorem, we prove the existence and uniqueness of a solution of the boundary value problem for system (2.1).

III. Let us fix p points $y^1, y^2, \dots, y^p$ in some neighborhood of a solution of the averaged problem and consider the following boundary value problem:

$$\begin{aligned} \dot{x} &= \varepsilon X(t,x), \ \ t \neq t_i(y^i), \\ \Delta x\big|_{t=t_i(y^i)} &= \varepsilon I_i(y^i), \\ F\Big(x(0), x\Big(\frac{T}{\varepsilon}\Big)\Big) &= 0. \end{aligned}$$

From what has been proved above, we conclude that this boundary value problem, for ε small enough, has a unique solution $x(t, y^1, \dots, y^p)$. If we choose $y^1, \dots, y^p$ so that

$$y^i = x(t_i(y^i), y^1, \dots, y^p), \ \ i = \overline{1,p}, \tag{2.3}$$

then the function $x(t, y^1, \dots, y^p)$ is the desired solution of problem (1.1). Using a fixed-point theorem, we show that system (2.3) has a solution. This completes the proof.

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Autonomous Linear Differential Equations with the Hukuhara Derivative that Preserve Polytopes

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By $\Omega(\mathbb{R}^d)$ we denote the set of all nonempty bounded subsets of $\mathbb{R}^d$. For the set of all nonempty compact subset of $\mathbb{R}^d$ we use the notation $K(\mathbb{R}^d)$. By $K_c(\mathbb{R}^d)$ we denote the subset of $K(\mathbb{R}^d)$ that consists of convex sets.

Definition 1. A set $C = A + B \overset{\text{def}}{=} \{a + b : \ a \in A, \ b \in B\}$ is called the *Minkowski sum* of two subsets $A, B \subset \mathbb{R}^d$.

Definition 2 ([1])**.** A set $C \in K_c(\mathbb{R}^d)$ is called the *Hukuhara difference* of $A, B \in K_c(\mathbb{R}^d)$, and denoted by $C = A - B$, if $A = B + C$.

Note that the Hukuhara difference $A - B$ is not defined for any pair of sets $A, B \in K_c(\mathbb{R}^d)$. Moreover, if there exists a set $C \in K_c(\mathbb{R}^d)$ such that $A = B + C$, then, generally speaking, $C \neq A + (-B)$. Indeed, let's take, for example, $A = [0, 2]$ and $B = [0, 1]$. For the set $C = [0, 1]$ we have $A = B + C$. At the same time, $A + (-B) = [-1, 2]$. The following theorem gives a necessary and sufficient condition for the existence of Hukuhara difference between sets $A, B \in K_c(\mathbb{R}^d)$.

Theorem 1 ([2, p. 8])**.** *Let $A, B \in K_c(\mathbb{R}^d)$ be convex compact sets. The Hukuhara difference $A - B$ exists if and only if for any boundary point $a \in \partial A$ there exists at least one point $c \in \mathbb{R}^d$ such that*

$$a \in B + \{c\} \subset A.$$

If the Hukuhara difference $A - B$ exists, then it is unique. This statement can be derived from the following lemma.

Lemma 1 ([2, p. 10])**.** *Let $C \subset \mathbb{R}^d$, $D \in K_c(\mathbb{R}^d)$, $B \in \Omega(\mathbb{R}^d)$, and $C + B \subset D + B$. Then $C \subset D$.*

By $\overline{\mathbb{B}}^d \overset{\text{def}}{=} \{x \in \mathbb{R}^d : \ \|x\| \le 1\}$ we denote the closed ball of unit radius centered at the origin.

Definition 3. *Hausdorff distance* $h(A, B)$ between sets A and $B \in \Omega(\mathbb{R}^d)$ is defined as

$$h(A, B) \overset{\text{def}}{=} \inf_{r \ge 0} \big\{r : \ A \subset B + r\overline{\mathbb{B}}^d, \ B \subset A + r\overline{\mathbb{B}}^d\big\}.$$

It follows directly from the definition that $h(A, B) = 0$ iff $\overline{A} = \overline{B}$. According to Hahn's theorem the pair $(K(\mathbb{R}^d), h)$ is a complete separable metric space, and $K_c(\mathbb{R}^d)$ is its closed subset.

By $I \subset \mathbb{R}$ we denote an arbitrary open interval that may be unbounded.

Definition 4. A map $X \colon I \to K_c(\mathbb{R}^d)$ is called *differentiable by Hukuhara* at $t_0 \in I$ if the limits

$$\lim_{\Delta t \to +0} \frac{X(t_0 + \Delta t) - X(t_0)}{\Delta t} \ \text{ and } \ \lim_{\Delta t \to +0} \frac{X(t_0) - X(t_0 - \Delta t)}{\Delta t}$$

both exist and are equal to the same convex compact set $D_H X(t_0)$, that is called *Hukuhara derivative* of X at t_0.

It is easy to see that if the map $X : I \to K_c(\mathbb{R}^d)$ is differentiable at every point of I, then for any $a < b \in I$ the difference $X(b) - X(a)$ is defined. Therefore, according to Theorem 1, non-decreasing of $\operatorname{diam} X(\,\cdot\,)$ is a necessary condition for the existence of Hukuhara derivative $D_H X(t)$, $t \in I$.

For a given positive integer $d \in \mathbb{N}$, let us consider the linear differential equation

$$D_H X(t) = A(t)X(t), \quad X(t) \in K_c(\mathbb{R}^d), \quad t \geq 0, \tag{1}$$

with a semi-continuous coefficient $d \times d$ matrix. By a polytope we mean a convex hull of finite number of points in $\mathbb{R}^d$.

Definition 5. We say that equation (1) *preserves polytopes* if for any its solution $X(\,\cdot\,)$ such that $X(0)$ is a polytope, it follows that $X(t)$ is a polytope for all $t \geq 0$.

Let us consider the problem of obtaining a necessary and sufficient condition for equation (1) to preserve polytopes. This problem is partially solved, namely, we obtained the complete description of the autonomous differential equations (1) that posses this property.

Theorem 2. *Equation* (1) *with a constant coefficient matrix $A(\,\cdot\,) \equiv A$ preserves polytopes if and only if there exists a real number λ and non-negative integers $a < b$ such that $A^b = \lambda A^a$.*

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Topological Entropy of a Diagonalizable Linear System of Differential Equations

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Consider the linear differential system

$$\dot{x} = A(t)x, \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}_+ \equiv [0, \infty), \tag{1}$$

with a piecewise continuous operator function $A : \mathbb{R}_+ \to \operatorname{End} \mathbb{R}^n$. Let us endow the space $\mathbb{R}^n$ with the norm $\|x\| = \max\limits_{1 \leqslant k \leqslant n} |x_k|$ and with the set of metrics

$$d_t^A(x_0, y_0) = \max_{\tau \in [0,t]} \|x(\tau, x_0) - x(\tau, y_0)\|, \quad x_0, y_0 \in \mathbb{R}^n, \quad t \in \mathbb{R}_+,$$

where $x(\,\cdot\,, a)$ is the solution to system (1) satisfying the condition $x(0, a) = a$. By $S_{\|\,\cdot\,\|}(A, \mathcal{K}, \varepsilon, t)$ we denote the *ε-entropy* of a compact metric space $\mathcal{K} \subset \mathbb{R}^n$ with the metric d_t^A [2] (that is, the minimum number of open balls of radius $\varepsilon > 0$ covering $\mathcal{K}$). Then the *topological entropy* [1] of system (1) is defined by the formula

$$h_{\rm top}(A) = \sup_{\mathcal{K} \subset \mathbb{R}^n} \lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} \, \frac{1}{t} \, \ln S_{\|\,\cdot\,\|}(A, \mathcal{K}, \varepsilon, t)$$

(its right-hand side does not depend on the choice of a norm $\|\cdot\|$, therefore the definition is correct).

In what follows we will use one more formula to calculate the topological entropy. For any $\varepsilon > 0$ and $n \in \mathbb{N}$ we denote by $N_{\|\,\cdot\,\|}(A, \mathcal{K}, \varepsilon, t)$ the *ε-capacity* of a compact metric space $\mathcal{K} \subset \mathbb{R}^n$ with the metric d_t^A [2] (i.e., the maximum number of points such that all their pairwise d_t^A-distances are greater than ε), then the topological entropy can be calculated by the formula

$$h_{\rm top}(A) = \sup_{\mathcal{K} \subset \mathbb{R}^n} \lim_{\varepsilon \to 0} \overline{\lim_{t \to \infty}} \, \frac{1}{t} \, \ln N_{\|\,\cdot\,\|}(A, \mathcal{K}, \varepsilon, t).$$

In [3], it is asserted that for the Lyapunov exponents $\lambda_1(A) \leqslant \cdots \leqslant \lambda_n(A)$ of any system (1) with a bounded operator function A, the equality

$$h_{\rm top}(A) = \sum_{\lambda_i(A) > 0} \lambda_i(A) \tag{2}$$

holds. In fact, it may not hold, as shown by

Theorem 1. *For system* (1) *with the operator function*

$$A(t) = \operatorname{diag}(a(t), b(t)), \quad \textit{where} \ \ (a(t), b(t)) = \begin{cases} (1,0), & t \in [0,1]; \\ (1,0), & t \in [(2n-1)!, (2n)!]; \\ (0,1), & t \in [(2n)!, (2n+1)!], \end{cases} \quad n = 1, 2, \dots, \tag{3}$$

relation (2) *becomes the inequality* $1 < 2$.

Proof. Let us calculate the Lyapunov exponents of system (3). On the one hand, from the inequalities $a(t) \leqslant 1$, $b(t) \leqslant 1$, $t \geq 0$, we see that the Lyapunov exponents do not exceed 1.

On the other hand, from the inequalities

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \int_0^t a(\tau)\,d\tau \geqslant \lim_{n\to+\infty} \frac{(2n)! - (2n-1)!}{(2n)!} = 1,$$

$$\overline{\lim_{t\to+\infty}} \frac{1}{t} \int_0^t b(\tau)\,d\tau \geqslant \lim_{n\to+\infty} \frac{(2n+1)! - (2n)!}{(2n+1)!} = 1,$$

it follows that the Lyapunov exponents of system (3) satisfy the equalities

$$\lambda_1(A) = \lambda_2(A) = 1 \ \text{ and } \ \lambda_1(A) + \lambda_2(A) = 2.$$

Let us calculate the topological entropy of system (3). For given $m, n \in \mathbb{N}$, consider the set of points of the form $\begin{pmatrix} x_1^{(k)} \\ 0 \end{pmatrix}$, where

$$x_1^{(k)} = \frac{k}{e^{(2n)!-(2n-1)!}m}\,, \quad k = 0, \dots, [e^{(2n)!-(2n-1)!}m]$$

($[\,\cdot\,]$ is an integer part of the number). Since the distance $d^A_{(2n)!}\Big(\begin{pmatrix} x_1^{(k)} \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^{(l)} \\ 0 \end{pmatrix}\Big)$, $k \neq l$, satisfies the inequality

$$d^A_{(2n)!}\Big(\begin{pmatrix} x_1^{(k)} \\ 0 \end{pmatrix}, \begin{pmatrix} x_1^{(l)} \\ 0 \end{pmatrix}\Big) = \frac{|k-l|}{e^{(2n)!-(2n-1)!}m}\, e^{(2n)!-(2n-1)!+(2n-2)!-(2n-3)!+\cdots} > \frac{1}{m}\,,$$

then

$$N\Big(A, [0,1] \times \{0\}, \frac{1}{m}\,, (2n)!\Big) \geqslant e^{(2n)!-(2n-1)!}m,$$

and hence

$$h_{\rm top}(A) \geqslant \lim_{n\to\infty} \Big(\frac{(2n)! - (2n-1)!}{(2n)!} + \frac{\ln m}{(2n)!}\Big) = 1.$$

Let us prove the opposite inequality $h_{\rm top}(A) \leqslant 1$. For an arbitrary compact set $K \subset \mathbb{R}^2$ we denote by γ_K a positive number such that $K \subset [-\gamma_K, \gamma_K] \times [-\gamma_K, \gamma_K]$.

For any $t > 0$ and $m \in \mathbb{N}$, the set of points $V_{t,m}$ of the form $\begin{pmatrix} x_1^{(k)} \\ x_2^{(l)} \end{pmatrix}$, where

$$x_1^{(k)} = \frac{k\gamma_K}{m}\, e^{-\int_0^t a(\tau)\,d\tau}, \quad k = -\big[e^{\int_0^t a(\tau)\,d\tau} m\big], \dots, \big[e^{\int_0^t a(\tau)\,d\tau} m\big],$$

$$x_2^{(l)} = \frac{l\gamma_K}{m} e^{-\int_0^t b(\tau)\,d\tau}, \quad l = -\big[e^{\int_0^t b(\tau)\,d\tau} m\big], \dots, \big[e^{\int_0^t b(\tau)\,d\tau} m\big],$$

is a $\frac{\gamma_K}{m}$-covering of the square $[-\gamma_K, \gamma_K] \times [-\gamma_K, \gamma_K]$. Indeed, let an arbitrary point $(x_1, x_2) \in [-\gamma_K, \gamma_K] \times [-\gamma_K, \gamma_K]$ be given. Then by the definition of the set $V_{t,m}$ there exists a point $(x_1^{(k_0)}, x_2^{(l_0)})$ such that

$$|x_1^{(k_0)} - x_1| \leqslant \frac{\gamma_K}{m}\, e^{-\int_0^t a(\tau)\,d\tau}, \quad |x_2^{(l_0)} - x_2| \leqslant \frac{\gamma_K}{m}\, e^{-\int_0^t b(\tau)\,d\tau}.$$

It follows that
$$|x_1^{(k_0)} - x_1| e^{\int\limits_0^t a(\tau)\,d\tau} \leqslant \frac{\gamma_K}{m}\,, \quad |x_2^{(l_0)} - x_2| e^{\int\limits_0^t b(\tau)\,d\tau} \leqslant \frac{\gamma_K}{m}\,.$$
Thus $S_{\|\,\cdot\,\|}(A,\mathcal{K},\varepsilon,t)$ does not exceed the cardinality of the set $V_{t,m}$, which equals
$$\Big(2\big[e^{\int\limits_0^t a(\tau)\,d\tau} m\big]+1\Big)\Big(2\big[e^{\int\limits_0^t b(\tau)\,d\tau} m\big]+1\Big) \leqslant 9m^2 e^{\int\limits_0^t (a(\tau)+b(\tau))\,d\tau} = 9m^2e^t,$$
whence we get $h_{\rm top}(A) \leqslant 1$. $\square$

For an arbitrary piecewise continuous function $a(\,\cdot\,):\mathbb{R}_+\to\mathbb{R}$, let
$$a^+(t) = \max_{s\in[0;t]} \int\limits_0^s a(\tau)\,d\tau.$$

Theorem 2. *If system* (1) *can be reduced to a diagonal form*
$$\dot y = \operatorname{diag}(b_1(t),\dots,b_n(t))y,$$
by means of a transformation $x = Q(t)y$ such that
$$\overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\ln\|Q(t)\| = \overline{\lim_{t\to+\infty}}\,\frac{1}{t}\,\ln\|Q^{-1}(t)\| = 0,$$
then
$$h_{\rm top}(A) = \overline{\lim_{t\to+\infty}}\,\frac{1}{t}\sum_{k=1}^n b^+(t).$$

Given a metric space $\mathcal{M}$ and a continuous map
$$A:\mathcal{M}\times\mathbb{R}_+\to \operatorname{End}\mathbb{R}^n \tag{4}$$
we form the function
$$\mu \longmapsto h_{\rm top}(A(\mu,\,\cdot\,)). \tag{5}$$
Results of [4, 5] imply

Theorem 3. *For any mapping* (4)*, function* (5) *belongs to the third Baire class. If $\mathcal{M}=[0,1]$, then for some mapping* (4) *function* (5) *does not belong to the first Baire class.*

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Coupled-Jumping Timescale Theory and Applications

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Abstract

Recently, the concept of a coupled-jumping timescale space (short for CJTS) $\mathbb{T}_1 - \mathbb{T}_2$ was initiated. Based on it, the theory of calculus and fundamental functions were established. By using this theory, an initial value problem of time-hybrid dynamic equations whose initial value is given in $\mathbb{T}_2$ and the unique solution is located in $\mathbb{T}_1$ can be considered. It is worth noting that the Hilger's theory can be derived through removing the coupled-jumping state by letting $\mathbb{T}_1 = \mathbb{T}_2$ and the Hilger theory is essentially based on a single timescale space. The coupled-jumping timescale theory largely deepens and includes the Hilger theory and brings a completely new significance of dynamic equations on time scales.

1 Vertical evolution of time scales

The time scale theory was introduced by Hilger in 1988 to unify the continuous and discrete analysis (see [1, 2]). This theory plays a very significant role in both pure and applied mathematics, for example, time-hybrid dynamic equations (see [11]), quaternion dynamic equations (see [3,4]), fuzzy dynamic equations (see [5]), the closedness of time scales and related function theory (see [6–9]), stochastic dynamic equations (see [10]) and hybrid measurability theory (see [12]), etc. To further reveal the changing essence of time scales, we first introduced two basic types of the evolution of time scales under which some corresponding dynamic equation were presented (see [11]).

In Figure 1, let $\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4\}$ be a timescale group. By Hilger theory, this time scale group will induce a continuous dynamic equation, a piecewise continuous dynamic equation, a discrete dynamic equation and a quantum dynamic equation in sequence. Starting with the evolution process of these time scales, $\mathbb{T}$ varies from the form $\mathbb{T}_1$ to the form $\mathbb{T}_4$ in the timescale group, such a **vertical evolution** in the timescale group acts as a direct factor which leads to the four different types of dynamic equation during the changing process of the time scale $\mathbb{T}$. Only when $\mathbb{T}$ is fixed in this timescale group, the concrete dynamic equation can be determined. From the viewpoint of the evolution process of time scales, the essence of Hilger's theory depends on the vertical evolution of time scales, accordingly the unification of various types of dynamic equation can be achieved when the form of $\mathbb{T}$ is fixed in a timescale group. In other words, the related analysis and applications on Hilger theory are purely based on a **single time scale** during this evolution.

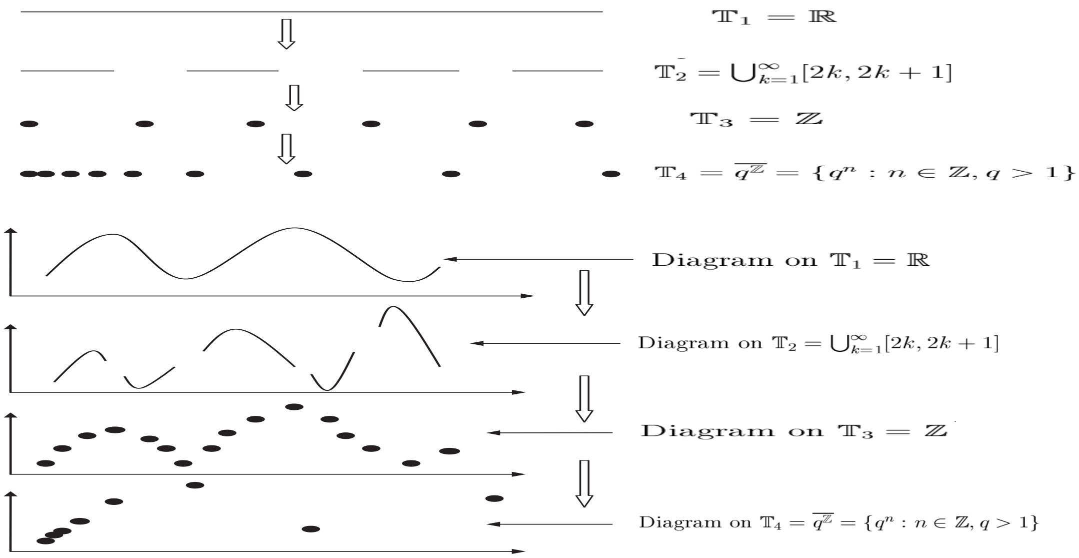

Figure 1. The vertical evolution diagram of dynamical behavior from $\mathbb{T}_1$ to $\mathbb{T}_4$ under Hilger theory

2 Hybrid-timescale problems-a horizontal evolution of time scales

The other natural and significant evolution of time scales that must be referred to is **horizontal evolution** of time scales. The related problems caused by horizontal evolution of time scales cannot be solved by Hilger theory and they still belong to the problems of timescale category. In Figure 2, let

$$\mathbb{T}_1 = \overline{\big\{q^n : q > 1, n \in \mathbb{Z}^- \cup \{0\}\big\}}, \quad \mathbb{T}_2 = [1.1, 3.7], \quad \mathbb{T}_3 = \bigcup_{k=2}^{5}[2k, 2k+1],$$
$$\mathbb{T}_4 = \{12.1, 13.1, 14.1, 15.1, 16.1\}, \quad \mathbb{T}_5 = \overline{\{(1.5)^n : n \geq 7\}}, \dots .$$

For convenience, let a timescale group be formed by $\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4, \mathbb{T}_5, \dots\}$. It is easy to observe that the dynamical behavior described by Figure 2 exists on the time scale $\mathbb{T}$ formed by five districts and each district is a time scale, i.e., $\mathbb{T} = \mathbb{T}_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3 \cup \mathbb{T}_4 \cup \mathbb{T}_5 \cup \dots$. Therefore, the switch of the dynamical behavior in four timescale districts is directly caused by a **horizontal evolution** of all the time scales in this timescale group.

Usually, all the similar problems described by Figures 2 are called the **hybrid-timescale problems**. Essentially, the hybrid-timescale problems are formed by the problems on multiple time scales and this class of problems can be precisely depicted by a **horizontal evolution** of time scales in a timescale group.

By comparison, the related hybrid-timescale problems are more comprehensive and will strictly include the problems on a single time scale as their particular cases (see Figure 3 for their detailed

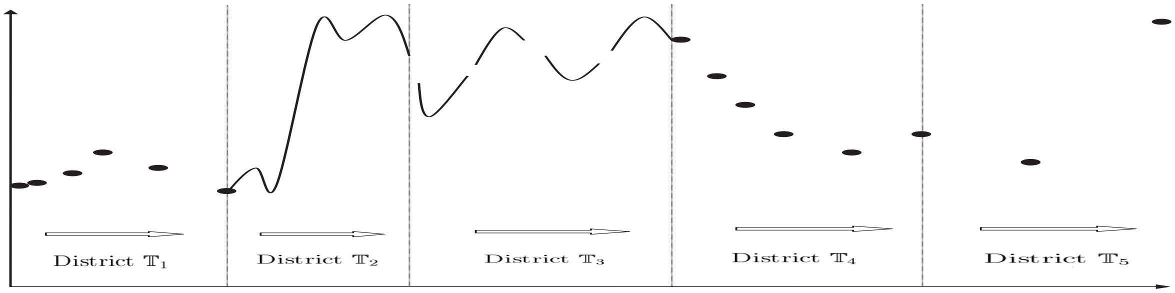

Figure 2. The horizontal evolution diagram of dynamical behavior from $\mathbb{T}_1$ to $\mathbb{T}_4$ under coupled-jumping timescale theory

relations). Moreover, the dynamical behavior on hybrid time scales cannot be effectively studied purely on a single time scale through Hilger theory. Therefore, it is very necessary to establish a theory (we call it coupled-jumping timescale theory) to solve the hybrid-timescale problems.

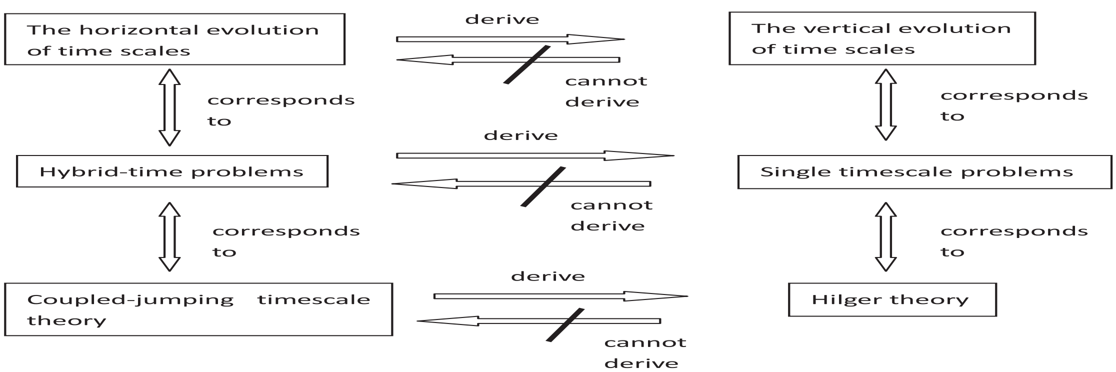

Figure 3. The relation among hybrid-timescale problems, single-timescale problems, Hilger theory and Coupled-jumping timescale theory

3 The description of the hybrid-timescale initial-value problems

For understanding the idea to solve the hybrid-timescale problems, we will adopt Figure 2 to illustrate our methods and the framework of the solving steps. Let a timescale group be $\{\mathbb{T}_1, \mathbb{T}_2, \mathbb{T}_3, \mathbb{T}_4, \mathbb{T}_5, \dots\}$. To break through the limitation of the Hilger theory and to establish a coupled-jumping timescale theory, demonstrating a distinct dynamical behavior on time scales, firstly, we must consider the formation process of the dynamical behavior in Figure 2. Assume that the dynamical behavior in Figure 2 corresponds to a solution $x(t)$ of a dynamic equation on the hybrid time scales with the initial point $(t_0, x(t_0))$, where $t_0 = 0 \in \mathbb{T}_1$. According to the continuous dependence on initial values of solutions and the continuation theorem, there is a solution on the district $\mathbb{T}_1$ such that $(t_1, x(t_1))$ is the right boundary point on the district $\mathbb{T}_1$, where $t_1 = 1 \notin \mathbb{T}_2$. Now taking $(t_1, x(t_1))$ as the initial point, there is a solution on the district $\mathbb{T}_2$ such that $(t_2, x(t_2))$ is the right boundary point on the district $\mathbb{T}_2$, where $t_2 = 3.7 \notin \mathbb{T}_3$. Next, by taking $(t_2, x(t_2))$ as the initial point, there is a solution on the district $\mathbb{T}_3$ such that $(t_3, x(t_3))$ is the right boundary point on the district $\mathbb{T}_3$, where $t_3 = 11 \notin \mathbb{T}_4$. Repeating the process, by taking $(t_3, x(t_3))$ as the initial point, there is a solution on the district $\mathbb{T}_4$ such that $(t_4, x(t_4))$ is the right boundary point on the district $\mathbb{T}_4$, where $t_4 = 16.1 \notin \mathbb{T}_5$. Finally, the solution on the district $\mathbb{T}_5$ is determined by the initial point $(t_4, x(t_4))$. If there are more time scales after $\mathbb{T}_5$, for instance, $\mathbb{T}_6, \mathbb{T}_7, \dots$, the process above can be continued until the solution exists on $\mathbb{T}_1 \cup \mathbb{T}_2 \cup \mathbb{T}_3 \cdots := \bigcup\limits_{i=1}^{+\infty} \mathbb{T}_i$.

In the above process, a key problem appears. Note that $t_1 \notin \mathbb{T}_2$ but the solution on district $\mathbb{T}_2$ is continuously dependent on $(t_1, x(t_1))$, similarly, $t_2 \notin \mathbb{T}_3$ but the solution on district $\mathbb{T}_3$ is continuously dependent on $(t_2, x(t_2))$,..., $t_4 \notin \mathbb{T}_5$ but the solution on district $\mathbb{T}_5$ is continuously dependent on $(t_4, x(t_4)), \dots$. Therefore, the first problem we must solve is that we should introduce an initial value problem of a dynamic equations whose initial value is given in one time scale and the unique solution is located in another. In the literature [11], the coupled-jumping timescale theory (or hybrid-timescale theory) was proposed.

4 The coupled-jumping timescale space (CJTS) and calculus

We present a notion of coupled-jumping timescale space and a concept of the hybrid-composition integral.

Definition 4.1 ([2])**.** For $\widehat{t} \in \mathbb{T}_k$, we define the forward jump operator $\sigma_k : \mathbb{T}_k \to \mathbb{T}_k$ by $\sigma_k(\widehat{t}) = \inf\{s \in \mathbb{T}_k : \ s > \widehat{t}\}$; the backward jump operator $\rho_k : \mathbb{T}_k \to \mathbb{T}_k$ by $\rho_k(\widehat{t}) = \sup\{s \in \mathbb{T}_k : \ s < \widehat{t}\}$; and the graininess function $\mu_k : \mathbb{T}_k \to [0, +\infty)$ by $\mu_k(\widehat{t}) = \sigma_k(\widehat{t}) - \widehat{t}$, where $k = 1, 2$.

Now, we will introduce the jumping construction of the coupled-jumping timescale space $\mathbb{T}_1 - \mathbb{T}_2$.

Definition 4.2 ([11])**.** Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be a pair of time scales. For $t \in \mathbb{T}_1 \cup \mathbb{T}_2$, we define the coupled-forward jump operator between $\mathbb{T}_1$ and $\mathbb{T}_2$ by $\sigma_{\mathbb{T}_2}(t) = \inf\{s \in \mathbb{T}_2 : \ s \geq t\}$, and define the coupled-backward jump operator between $\mathbb{T}_1$ and $\mathbb{T}_2$ by $\rho_{\mathbb{T}_2}(t) = \sup\{s \in \mathbb{T}_2 : \ s \leq t\}$. We say t is a coupled right-dense point iff $\sigma_{\mathbb{T}_2}(t) = t$; t is a coupled right-scattered point iff $\sigma_{\mathbb{T}_2}(t) > t$; t is a coupled left-dense point iff $\rho_{\mathbb{T}_2}(t) = t$; t is a coupled left-scattered point iff $\rho_{\mathbb{T}_2}(t) < t$; t is a coupled isolated point iff $\rho_{\mathbb{T}_2}(t) < t < \sigma_{\mathbb{T}_2}(t)$ (see Figure 4).

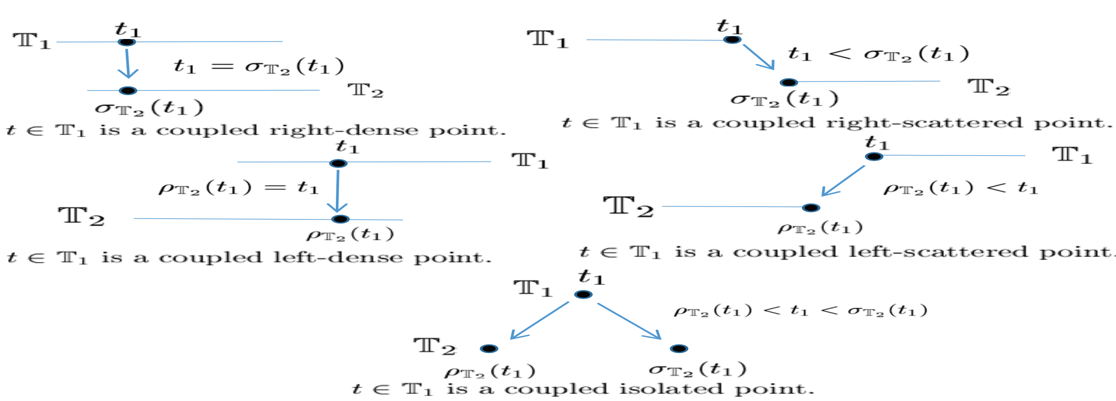

Figure 4. Schematic diagram of all types of coupled-jumping points

Definition 4.3 ([11])**.** Let $f : \mathbb{T}_1 \cup \mathbb{T}_2 \to \mathbb{R}$. We define a hybrid-composition integral (or short for

HC-integral) of $f(t)$ on CJTS as follows:

$$\int\limits_a^b f(\tau)\Delta_m\tau = \begin{cases} \alpha \int [\sigma_{\mathbb{T}_1}(a), \rho_{\mathbb{T}_1}(b)]_{\mathbb{T}_1} f(\tau)\Delta_1\tau + (1-\alpha) \int\limits_{[\sigma_{\mathbb{T}_2}(a), \rho_{\mathbb{T}_2}(b)]_{\mathbb{T}_2}} f(\tau)\Delta_2\tau, & a < b, \\ -\alpha \int\limits_{[\sigma_{\mathbb{T}_1}(b), \rho_{\mathbb{T}_1}(a)]_{\mathbb{T}_1}} f(\tau)\Delta_1\tau - (1-\alpha) \int [\sigma_{\mathbb{T}_2}(b), \rho_{\mathbb{T}_2}(a)]_{\mathbb{T}_2} f(\tau)\Delta_2\tau, & a > b, \end{cases}$$

where $a, b \in \mathbb{T}_1 \cup \mathbb{T}_2$, $0 \le \alpha \le 1$ and α is called the hybrid-composition proportion coefficient.

5 Time-hybrid dynamic equations on CJTS

In this section, we will introduce the exponential function on coupled-jumping time scales and introduce the basic theorem of time-hybrid dynamic equations. For more details, one may consult the literature [11].

Definition 5.1 ([11])**.** Let $\ddot{t}, s \in \mathbb{T}_1 \cup \mathbb{T}_2$. We introduce the HC-exponential function by

$$\overline{e}_f(\ddot{t}, s) := \begin{cases} \exp\Bigg\{\alpha \int\limits_{[\sigma_{\mathbb{T}_1}(s), \rho_{\mathbb{T}_1}(\ddot{t})]_{\mathbb{T}_1}} \dfrac{\operatorname{Log}(1+\mu_1(\tau)f(\tau))}{\mu_1(\tau)}\,\Delta_1\tau \\ \qquad\qquad + (1-\alpha) \int\limits_{[\sigma_{\mathbb{T}_2}(s), \rho_{\mathbb{T}_2}(\ddot{t})]_{\mathbb{T}_2}} \dfrac{\operatorname{Log}(1+\mu_2(\tau)f(\tau))}{\mu_2(\tau)}\,\Delta_2\tau\Bigg\}, & s < \ddot{t}, \\ \exp\Bigg\{-\alpha \int\limits_{[\sigma_{\mathbb{T}_1}(\ddot{t}), \rho_{\mathbb{T}_1}(s)]_{\mathbb{T}_1}} \dfrac{\operatorname{Log}(1+\mu_1(\tau)f(\tau))}{\mu_1(\tau)}\,\Delta_1\tau \\ \qquad\qquad - (1-\alpha) \int\limits_{[\sigma_{\mathbb{T}_2}(\ddot{t}), \rho_{\mathbb{T}_2}(s)]_{\mathbb{T}_2}} \dfrac{\operatorname{Log}(1+\mu_2(\tau)f(\tau))}{\mu_2(\tau)}\,\Delta_2\tau\Bigg\}, & s > \ddot{t}. \end{cases}$$

In the following theorem, we will demonstrate the HC-exponential solution of the homogeneous time-hybrid dynamic equation.

Theorem 5.1 ([11])**.** *Let $t \in \mathbb{T}_1^{\overline{\kappa}}$, $s \in \mathbb{T}_2^{\overline{\kappa}}$, $t \ge s$. Then $\overline{e}_f(t,s)$ is the solution of the initial value problem*

$$\mu_1(t)x^{\Delta_t}(t) = \Bigg\{\big(1+\mu_1(t)f(t)\big)^\alpha \exp\Bigg\{(1-\alpha) \int\limits_{\rho_{\mathbb{T}_2}(t)}^{\rho_{\mathbb{T}_2}(\sigma_1(t))} \frac{\operatorname{Log}(1+\mu_2(\tau)f(\tau))}{\mu_2(\tau)}\,\Delta_2\tau\Bigg\} - 1\Bigg\} x(t), \tag{5.1}$$

with the initial value $x(s) = 1$, where $x^{\Delta_t}(t)$ denotes the Δ-derivative at t on $\mathbb{T}_1$.

Remark 5.1. Notice that the initial value problem of the homogeneous time-hybrid dynamic equation (5.1) has the characteristic that the initial value is given in $\mathbb{T}_2$ and the unique solution is located in $\mathbb{T}_1$, where $\mathbb{T}_1$ **may not be equal to** $\mathbb{T}_2$. There has been no theory to support the study of such a type of time-hybrid dynamic equation before now.

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