Set of Points of Lower Semicontinuity for the Topological Entropy of a Family of Dynamical Systems Continuously Depending on a Parameter

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Let us give a precise definition of topological entropy [1]. Let (X,d) be a compact metric space and let $f:X\to X$ be a continuous mapping. Along with the original metric d, we define an additional system of metrics

$$d_n^f(x,y)=\max_{0\le i\le n-1} d(f^i(x),f^i(y)),\ \ x,y\in X,\ \ n\in\mathbb{N},$$

where f^i, $i\in\mathbb{N}$, is the i-th iteration of the mapping f, $f^0\equiv \mathrm{id}_X$. For any $n\in\mathbb{N}$ and $\varepsilon>0$, by $N_d(f,\varepsilon,n)$ we denote the maximum number of points in X such that the pairwise d_n^f-distances between them are greater than ε. Such a set of points is said to be (f,ε,n)-separated. Then the *topological entropy* of the dynamical system generated by the continuous mapping f is defined as the quantity (which may be a nonnegative real number or infinity)

$$h_{\mathrm{top}}(f)=\lim_{\varepsilon\to 0}\overline{\lim_{n\to\infty}}\,\frac{1}{n}\,\ln N_d(f,\varepsilon,n). \tag{1}$$

Note that the quantity (1) remains unchanged if the metric d in its definition is replaced by any other metric that defines the same topology on X as d; this, in particular, explains why the entropy (1) is said to be topological.

Given a metric space $\mathcal{M}$ and a jointly continuous mapping

$$f:\mathcal{M}\times X\to X \tag{2}$$

we form the function

$$\mu\longmapsto h_{\mathrm{top}}(f(\mu,\,\cdot\,)). \tag{3}$$

Recall that a point μ_0 of the metric space $\mathcal{M}$ is called a point of lower semicontinuity of a function $h:\mathcal{M}\to\mathbb{R}\cup\{\infty\}$ if for each sequence $(\mu_k)_{k\in\mathbb{N}}$ of points in $\mathcal{M}$ converging to μ_0, one has the inequality

$$h(\mu_0)\le \underline{\lim_{k\to+\infty}}\ h(\mu_k).$$

It was proved in [3] that if the space $\mathcal{M}$ is complete, then the property of lower semicontinuity is Baire typical for the topological entropy of a family of mappings (2); in other words, the set of points of $\mathcal{M}$ at which the function (3) is lower semicontinuous contains a dense G_δ-set in $\mathcal{M}$. It was established in [4] that the set of points of lower semicontinuity is itself an everywhere dense G_δ-set in $\mathcal{M}$. In addition, an example of a mapping (2) (where the parameter space $\mathcal{M}$ is the Cantor perfect set in the interval $[0,1]$) for which the set of points of lower semicontinuity is not an F_σ-set was constructed in [4].

By definition [2, p. 277], a metric space has dimension zero if any of its points has an arbitrarily small neighborhood that is simultaneously closed and open, which is equivalent to the emptiness of the boundary of this neighborhood. One example of such a space is the Cantor perfect set $\mathcal{K}$ (the set of infinite ternary fractions $x = 0,\ a_1a_2a_3, \dots$, where $a_i \in \{0,\ 2\}$) in the interval $[0, 1]$ with the metric induced by the natural metric of the real line.

A natural question arises: what is the set of lower semicontinuity points of a function (3). In the paper [5] we derived a complete description of the set of points of lower semicontinuity of a function (3) for each complete metric separable zero-dimensional space $\mathcal{M}$.

For an open everywhere dense subset of a complete metric separable zero-dimensional space $\mathcal{M}$ the following theorem holds.

Theorem 1. *Let $\mathcal{M}$ be a complete separable zero-dimensional metric space and let $X = \mathcal{K}$ be the Cantor perfect set in the interval $[0, 1]$ with the metric induced by the natural metric of the real line. Then for each open everywhere dense subset G of the space $\mathcal{M}$ there exists a mapping* (2) *such that the function* (3) *is bounded and its set of points of lower semicontinuity coincides with the set G.*

For an open everywhere dense G_δ-subset of a complete metric separable zero-dimensional space $\mathcal{M}$ the following theorem holds.

Theorem 2. *Let $\mathcal{M}$ be a complete separable zero-dimensional metric space and let $X = \mathcal{K}$ be the Cantor perfect set in the interval $[0, 1]$ with the metric induced by the natural metric of the real line. Then for each everywhere dense G_δ-subset G of the space $\mathcal{M}$ there exists a mapping* (2) *such that the set of points of lower semicontinuity of the function* (3) *coincides with the set G.*

References

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