Asymptotic Analysis of Two-Dimensional Cyclic Systems of First Order Nonlinear Differential Equations

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1 Introduction

This paper is concerned with positive solutions of the two-dimensional cyclic systems of first order nonlinear differential equations of the forms

(A) $\quad x' + p(t)y^{\alpha} = 0,\ y' - q(t)x^{\beta} = 0,\ t \geqq a;$

(B) $\quad x' - p(t)y^{\alpha} = 0,\ y' + q(t)x^{\beta} = 0,\ t \geqq a$

for which the following conditions are always assumed to hold:

(a) α and β are positive constants such that $\alpha\beta < 1$;

(b) $p, q : [a,\infty) \to (0,\infty)$, $a \geqq 0$ are regularly varying functions such that

$$p(t) = t^{\lambda} l(t), \quad q(t) = t^{\mu} m(t), \quad l, m \in \mathrm{SV}.$$

By a positive solution of (A) or (B) we mean a vector function $(x(t), y(t))$ both components of which are positive and satisfy the system (A) or (B) in a neighborhood of infinity. In this paper we are concerned with exclusively with positive solutions of (A) and (B) both components of which are regularly varying functions in the sense of Karamata. Such a solution $(x(t), y(t))$ is called regularly varying of index (ρ,σ) if $x(t)$ and $y(t)$ are regularly varying of indices $\rho(\in \mathbb{R})$ and $\sigma(\in \mathbb{R})$, respectively, and is denoted by $(x, y) \in \mathrm{RV}(\rho,\sigma)$.

Since the publication of the book [3] of Marić in the year 2000, the class of regularly varying functions in the sense of Karamata is a well-suited framework for the asymptotic analysis of nonoscillatory solutions of second order linear differential equation of the form

$$x'' = q(t)x, \quad q(t) > 0.$$

The definitions and properties of regularly varying functions

Definition 1.1. A measurable function $f : [a,\infty) \to (0,\infty)$ is said to be a regularly varying of index ρ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \ \text{ for any } \ \lambda > 0, \quad \rho \in \mathbb{R}.$$

Propsoition 1.1 (Representation Theorem)**.** *A measurable function $f : [a,\infty) \to (0,\infty)$ is regularly varying of index ρ if and only if it can be written in the form*

$$f(t) = c(t) \exp\bigg\{ \int\limits_{t_0}^{t} \frac{\delta(s)}{s}\, ds \bigg\}, \quad t \geqq t_0,$$

for some $t_0 > a$, where $c(t)$ and $\delta(t)$ are measurable functions such that

$$\lim_{t\to\infty} c(t) = c \in (0,\infty) \ \ \textit{and} \ \ \lim_{t\to\infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. The symbol SV is used to denote $\mathrm{RV}(0)$ and a member of $\mathrm{SV} = \mathrm{RV}(0)$ is referred to as a slowly varying function. If $f \in \mathrm{RV}(\rho)$, then $f(t) = t^{\rho} L(t)$ for some $L \in \mathrm{SV}$. Therefore, the class of slowly varying functions is of fundamental importance in the theory of regular variation. In addition to the functions tending to positive constants as $t \to \infty$, the following functions

$$\prod_{i=1}^{N} (\log_i t)^{m_i} \ \ (m_i \in \mathbb{R}), \quad \exp\Big\{\prod_{i=1}^{N} (\log_i t)^{n_i}\Big\} \ \ (0 < n_i < 1), \quad \exp\Big\{\frac{\log t}{\log_2 t}\Big\},$$

where $\log_1 t = \log t$ and $\log_k t = \log \log_{k-1} t$ for $k = 2, 3, \dots, N$, also belong to the set of slowly varying functions.

Propsoition 1.2. *Let $L(t)$ be any slowly varying function. Then, for any $\gamma > 0$,*

$$\lim_{t\to\infty} t^{\gamma} L(t) = \infty \ \ \textit{and} \ \ \lim_{t\to\infty} t^{-\gamma} L(t) = 0.$$

For the most complete exposition of the theory of regular variation and its applications the reader is referred to the book of Bingham, Goldie and Teugels [1].

2 Main results

The papers [2] and [4] are devoted to the analysis of strongly decreasing and increasing regularly varying solutions $(x, y) \in \mathrm{RV}(\rho, \sigma)$ of the system

(C) $\quad x' + p(t) y^{\alpha} = 0, \ y' + q(t) x^{\beta} = 0, \ t \geqq a;$

(D) $\quad x' - p(t) y^{\alpha} = 0, \ y' - q(t) x^{\beta} = 0, \ t \geqq a.$

(More precisely, $\rho < 0$ and $\sigma < 0$, $\rho = 0$ and $\sigma < 0$, $\rho < 0$ and $\sigma = 0$ for system (C), moreover, $\rho > 0$ and $\sigma > 0$, $\rho = 0$ and $\sigma > 0$, $\rho > 0$ and $\sigma = 0$ for system (D).) The purpose of this talk is to supplement necessary and sufficient conditions for the existence of regularly varying solutions $(x, y) \in \mathrm{RV}(\rho, \sigma)$ of (A) and (B) with either $\rho = 0$ or $\sigma = 0$, in which case either $x(t)$ or $y(t)$ is slowly varying, and then to determine their asymptotic behavior as $t \to \infty$ accurately. Our main results are the following.

Theorem 2.1. *System* (A) *possesses regularly varying solutions $(x, y) \in \mathrm{RV}(\rho, \sigma)$ with $\rho = 0$ and $\sigma > 0$ such that $\lim\limits_{t\to\infty} x(t) = 0$ and $\lim\limits_{t\to\infty} y(t) = \infty$ if and only if*

$$\lambda + 1 + \alpha(\mu + 1) = 0, \ \ \mu + 1 > 0$$

and

$$\int_a^{\infty} p(t) (t q(t))^{\alpha} \, dt < \infty,$$

in which case $\sigma = \mu + 1$ *and any such solution* $(x(t), y(t))$ *of* (A) *has one and the same asymptotic behavior*

$$x(t) \sim \bigg[(1-\alpha\beta)\int\limits_t^\infty p(s)\Big(\frac{sq(s)}{\sigma}\Big)^\alpha\,ds\bigg]^{\frac{1}{1-\alpha\beta}},\ \ t \to \infty,$$

$$y(t) \sim \frac{tq(t)}{\sigma}\bigg[(1-\alpha\beta)\int\limits_t^\infty p(s)\Big(\frac{sq(s)}{\sigma}\Big)^\alpha\,ds\bigg]^{\frac{\beta}{1-\alpha\beta}},\ \ t \to \infty,$$

where the symbol $\sim$ *is used to denote the asymptotic equivalence*

$$f(t) \sim g(t)\ \ as\ \ t \to \infty \iff \lim_{t\to\infty}\frac{f(t)}{g(t)} = 1.$$

Theorem 2.2. *System* (A) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho < 0$ *and* $\sigma = 0$ *such that* $\lim\limits_{t\to\infty} x(t) = 0$ *and* $\lim\limits_{t\to\infty} y(t) = \infty$ *if and only if*

$$\lambda + 1 < 0,\ \ \beta(\lambda + 1) + \mu + 1 = 0$$

and

$$\int\limits_a^\infty (tp(t))^\beta q(t)\,dt = \infty,$$

in which case $\rho = \lambda + 1$ *any such solution* $(x(t), y(t))$ *of* (A) *has one and the same asymptotic behavior*

$$x(t) \sim -\frac{tp(t)}{\rho}\bigg[(1-\alpha\beta)\int\limits_a^t \Big(\frac{sp(s)}{-\rho}\Big)^\beta q(s)\,ds\bigg]^{\frac{\alpha}{1-\alpha\beta}},\ \ t \to \infty,$$

$$y(t) \sim \bigg[(1-\alpha\beta)\int\limits_a^t \Big(\frac{sp(s)}{-\rho}\Big)^\beta q(s)\,ds\bigg]^{\frac{1}{1-\alpha\beta}},\ \ t \to \infty.$$

Theorem 2.3. *System* (A) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho < 0$ *and* $\sigma > 0$ *such that* $\lim\limits_{t\to\infty} x(t) = 0$ *and* $\lim\limits_{t\to\infty} y(t) = \infty$ *if and only if*

$$\lambda + 1 + \alpha(\mu + 1) < 0,\ \ \beta(\lambda + 1) + \mu + 1 > 0,$$

in which case

$$\rho = \frac{\lambda + 1 + \alpha(\mu + 1)}{1 - \alpha\beta},\ \ \sigma = \frac{\beta(\lambda + 1) + \mu + 1}{1 - \alpha\beta}$$

and any such solution $(x(t), y(t))$ *of* (A) *has one and the same asymptotic behavior*

$$x(t) \sim \Big[\frac{t^{1+\alpha}p(t)q(t)^\alpha}{-\rho\sigma^\alpha}\Big]^{\frac{1}{1-\alpha\beta}},\quad y(t) \sim \Big[\frac{t^{1+\beta}p(t)^\beta q(t)}{(-\rho)^\beta\sigma}\Big]^{\frac{1}{1-\alpha\beta}},\ \ t \to \infty.$$

Theorem 2.4. *System* (B) *possesses regularly varying solutions* $(x, y) \in \mathrm{RV}(\rho, \sigma)$ *with* $\rho = 0$ *and* $\sigma < 0$ *such that* $\lim\limits_{t\to\infty} x(t) = \infty$ *and* $\lim\limits_{t\to\infty} y(t) = 0$ *if and only if*

$$\lambda + 1 + \alpha(\mu + 1) = 0,\ \ \mu + 1 < 0$$

and

$$\int_a^\infty p(t)(tq(t))^\alpha\,dt = \infty,$$

in which case $\sigma = \mu + 1$ and any such solution $(x(t), y(t))$ of (B) *has one and the same asymptotic behavior*

$$x(t) \sim \bigg[(1-\alpha\beta)\int_a^t p(s)\Big(\frac{sq(s)}{-\sigma}\Big)^\alpha\,ds\bigg]^{\frac{1}{1-\alpha\beta}},\ \ t\to\infty,$$

$$y(t) \sim -\frac{tq(t)}{\sigma}\,\bigg[(1-\alpha\beta)\int_a^t p(s)\Big(\frac{sq(s)}{-\sigma}\Big)^\alpha\,ds\bigg]^{\frac{\beta}{1-\alpha\beta}},\ \ t\to\infty.$$

Theorem 2.5. *System* (B) *possesses regularly varying solutions $(x, y) \in \mathrm{RV}(\rho, \sigma)$ with $\rho > 0$ and $\sigma = 0$ such that $\lim\limits_{t\to\infty} x(t) = \infty$ and $\lim\limits_{t\to\infty} y(t) = 0$ if and only if*

$$\lambda + 1 > 0,\ \ \beta(\lambda+1) + \mu + 1 = 0$$

and

$$\int_a^\infty (tp(t))^\beta q(t)\,dt < \infty,$$

in which case $\rho = \lambda + 1$ and any such solution $(x(t), y(t))$ of (B) *has one and the same asymptotic behavior*

$$x(t) \sim \frac{tp(t)}{\rho}\,\bigg[(1-\alpha\beta)\int_t^\infty \Big(\frac{sp(s)}{\rho}\Big)^\beta q(s)\,ds\bigg]^{\frac{\alpha}{1-\alpha\beta}},\ \ t\to\infty,$$

$$y(t) \sim \bigg[(1-\alpha\beta)\int_t^\infty \Big(\frac{sp(s)}{\rho}\Big)^\beta q(s)\,ds\bigg]^{\frac{1}{1-\alpha\beta}},\ \ t\to\infty.$$

Theorem 2.6. *System* (B) *possesses regularly varying solutions $(x, y) \in \mathrm{RV}(\rho, \sigma)$ with $\rho > 0$ and $\sigma < 0$ such that $\lim\limits_{t\to\infty} x(t) = \infty$ and $\lim\limits_{t\to\infty} y(t) = 0$ if and only if*

$$\lambda + 1 + \alpha(\mu+1) > 0,\ \ \beta(\lambda+1) + \mu + 1 < 0,$$

in which case

$$\rho = \frac{\lambda+1+\alpha(\mu+1)}{1-\alpha\beta},\quad \sigma = \frac{\beta(\lambda+1)+\mu+1}{1-\alpha\beta}$$

and any such solution $(x(t), y(t))$ of (B) *has one and the same asymptotic behavior*

$$x(t) \sim \Big[\frac{t^{1+\alpha}p(t)q(t)^\alpha}{\rho(-\sigma)^\alpha}\Big]^{\frac{1}{1-\alpha\beta}},\quad y(t) \sim \Big[\frac{t^{1+\beta}p(t)^\beta q(t)}{-\rho^\beta\sigma}\Big]^{\frac{1}{1-\alpha\beta}},\ \ t\to\infty.$$

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References

[1] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation.* Encyclopedia of Mathematics and its Applications, 27. Cambridge University Press, Cambridge, 1989.

[2] J. Jaroš and T. Kusano, Existence and precise asymptotic behavior of strongly monotone solutions of systems of nonlinear differential equations. *Differ. Equ. Appl.* **5** (2013), no. 2, 185–204.

[3] V. Marić, *Regular Variation and Differential Equations.* Lecture Notes in Mathematics, 1726. Springer-Verlag, Berlin, 2000.

[4] T. Tanigawa, Existence and asymptotic behavior of strongly monotone solutions of nonlinear differential equations. *Differ. Equ. Appl.* **7** (2015), no. 1, 79–91.