The Equation in Variations for the Controlled Differential Equation with Delay and its Application

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The controlled differential equations with delay arise in different areas of natural sciences and economics. To illustrate this, below we will consider the simplest model of economic growth. Let $p(t)$ be a quantity of a product produced at the moment t expressed in money units. The fundamental principle of the economic growth has the form

$$p(t)=a(t)+i(t), \tag{1}$$

where $a(t)$ is the so-called apply function and $i(t)$ is a quantity induced investment. We consider the case where the functions $a(t)$ and $i(t)$ have the form

$$a(t)=u_1(t)p(t) \tag{2}$$

and

$$i(t)=u_2(t)p(t-\tau)+\alpha\dot{p}(t), \tag{3}$$

where $u_i(t)\in(0,1)$ for $i=1,2$, are control functions, $\alpha>0$ is a given number and $\tau>0$ is so-called delay parameter.

Formula (3) shows that the value of investment at the moment t depends on the quantity of money at the moment $t-\tau$ (in the past) and on the velocity (production current) at the moment t. From formulas (1)–(3) we get the delay controlled differential equation

$$\dot{p}(t)=\frac{1-u_1(t)}{\alpha}\,p(t)-\frac{u_2(t)}{\alpha}\,p(t-\tau). \tag{4}$$

Let $I=[t_0,t_1]$ be a given interval, suppose that $O\subset\mathbb{R}^n$ is an open set and $U\subset\mathbb{R}^r$ is a compact set. Let the n-dimensional function $f(t,x,y,u,v)$ be continuous on $I\times O^2\times U^2$ and continuously differentiable with respect to x, y and u, v. Furthermore, let $\tau_2>\tau_1>0$ and $\theta>0$

be given numbers; let Φ be a set of continuously differentiable functions $\varphi : I_1 = [\widehat{\tau}, t_0] \to O$, where $\widehat{\tau} = t_0 - \tau_2$ and let Ω be a set of piecewise-continuous functions $u(t) \in U$, $t \in I_2 = [\widehat{\theta}, t_1]$, where $\widehat{\theta} = t_0 - \theta$. To each element $\mu = (\tau, \varphi, u) \in \Lambda := [\tau_1, \tau_2] \times \Phi \times \Omega$ we assign the delay controlled differential equation

$$\dot{x}(t) = f\big(t, x(t), x(t-\tau), u(t), u(t-\theta)\big), \quad t \in (t_0, t_1) \tag{5}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in I_1. \tag{6}$$

Definition. Let $\mu = (\tau, \varphi, u) \in \Lambda$. A function $x(t; \mu) \in O$ for $t \in I_3 = [\widehat{\tau}, t_1]$, is called a solution of equation (5) with the initial condition (6), or a solution corresponding to the element μ and defined on the interval I_3, if $x(t; \mu)$ satisfies condition (6), is absolutely continuous on the interval I and it satisfies equation (5) almost everywhere on (t_0, t_1).

Let us introduce notations

$$|\mu| = |\tau| + \|\varphi\|_1 + \|u\|, \quad \Lambda_\varepsilon(\mu_0) = \big\{\mu \in \Lambda : \ |\mu - \mu_0| \le \varepsilon\big\},$$

where

$$\|\varphi\|_1 = \sup\big\{|\varphi(t)| + |\dot{\varphi}(t)| : \ t \in I_1\big\}, \quad \|u\| = \sup\big\{|u(t)| : \ t \in I_2\big\},$$

$\varepsilon > 0$ is a fixed number and $\mu_0 = (\tau_0, \varphi_0, u_0) \in \Lambda$ is a fixed initial element; furthermore,

$$\delta\tau = \tau - \tau_0, \ \ \delta\varphi(t) = \varphi(t) - \varphi_0(t), \ \ \delta u(t) = u(t) - u_0(t),$$
$$\delta\mu = \mu - \mu_0 = (\delta\tau, \delta\varphi, \delta u), \ \ |\delta\mu| = |\delta\tau| + \|\delta\varphi\|_1 + \|\delta u\|.$$

Theorem. *Let $x_0(t) := x(t; \mu_0)$ be the solution corresponding to the initial element $\mu_0 = (\tau_0, \varphi_0, u_0) \in \Lambda$ and defined on the interval I_3, where $\tau_0 \in (\tau_1, \tau_2)$. Then, there exists $\varepsilon_1 > 0$ such that for each perturbed element $\mu \in \Lambda_{\varepsilon_1}(\mu_0)$ there corresponds the solution $x(t; \mu)$ defined on the interval I_3 and the following representation holds*

$$x(t; \mu) = x_0(t) + \delta x(t; \delta\mu) + o(t; \delta\mu), \quad t \in (t_0, t_1), \tag{7}$$

where

$$\lim_{|\delta\mu| \to 0} \frac{|o(t; \delta\mu)|}{|\delta\mu|} = 0 \ \ \textit{uniformly for} \ \ t \in (t_0, t_1).$$

Moreover, the function

$$\delta x(t) = \begin{cases} \delta\varphi(t), & t \in I_1, \\ \delta x(t; \delta\mu), & t \in (t_0, t_1) \end{cases}$$

is a solution to the "equation in variations"

$$\dot{\delta} x(t) = f_x[t]\delta x(t) + f_y[t]\delta x(t - \tau_0) - f_y[t]\dot{x}_0(t - \tau_0)\delta\tau + f_u[t]\delta u(t) + f_v[t]\delta u(t - \theta), \quad t \in (t_0, t_1) \tag{8}$$

with the initial condition

$$\delta x(t) = \delta\varphi(t), \quad t \in [\widehat{\tau}, t_0]. \tag{9}$$

Here $f_x[t] = f_x(t, x_0(t), x_0(t - \tau_0), u_0(t), u_0(t - \theta))$.

The theorem is proved by the scheme given in [1]. Formula (7) and equation (8) allow us to obtain an approximate solution of the perturbed equation (5) in analytical form. In fact, for a small $|\delta\mu|$, from (7) it follows that

$$x(t;\mu) \approx x_0(t) + \delta x(t;\delta\mu), \quad t \in (t_0, t_1). \tag{10}$$

For the economical model (4), where $u_0(t) = (u_{10}(t), u_{20}(t))$ in the initial element $\mu_0 = (\tau_0, \varphi_0, u_0)$ and $p_0(t) = p(t;\mu_0)$, the equation in variations and the initial condition, respectively, have the forms

$$\begin{aligned}\dot{\delta p}(t) = \frac{1-u_{10}(t)}{\alpha}\,\delta p(t) - \frac{u_{20}(t)}{\alpha}\,\delta p(t-\tau_0)\\ + \frac{u_{20}(t)}{\alpha}\,\dot p_0(t-\tau_0)\delta\tau - \frac{p_0(t)}{\alpha}\,\delta u_1(t) - \frac{p_0(t-\tau_0)}{\alpha}\,\delta u_2(t), \quad t \in (t_0, t_1)\end{aligned}$$

and

$$\delta p(t) = \delta\varphi(t), \quad t \in [\widehat{\tau}, t_0].$$

Below, on the basis of formula (10) an approximate solution is constructed for the perturbed equation.

Example.

(a) Let $t_0 = 0$, $t_1 = 2$, $\tau_1 = 0.5$, $\tau_2 = 1.5$, $\tau_0 = 1$, $\varphi_0(t) \equiv 1$,

$$u_0(t) = \begin{cases} \sqrt{2(t+1)^2+1}, & t \in [0,1],\\ \sqrt{2(t+1)^2+t^2}, & t \in [1,2], \end{cases}$$

i.e., in this case $\mu_0 = (1, 1, u_0)$. Consider the scalar original equation

$$\dot x(t) = 2x^2(t) + x^2(t-1) - u_0^2(t) + 1, \quad t \in (0,2),$$

with the initial condition

$$x(t) = 1, \quad t \in [-1.5, 0].$$

It is easy to see that

$$x_0(t) := x(t;\mu_0) = \begin{cases} 1, & t \in [-1.5, 0],\\ t+1, & t \in [0,2]. \end{cases}$$

(b) The perturbed equation

$$\dot x(t) = 2x^2(t) + x^2(t-1-\rho_1) - [u_0(t) + \rho_3\ \sin(t)]^2 + 1, \quad t \in (0,2),$$

with the perturbed initial condition

$$x(t) = 1 + 2\rho_2\cos(t), \quad t \in [-1.5, 0],$$

where $|\rho_i|$ for $i = 1, 2, 3$ are small fixed numbers. In this case we have

$$\begin{gathered}\mu = (1+\rho_1, 1+2\rho_2\cos(t), u_0(t) + \rho_3\sin(t)),\\ \delta\tau = \rho_1, \delta\varphi(t) = 2\rho_2\cos(t), \quad \delta u(t) = \rho_3\sin(t).\end{gathered}$$

(c) It is clear that

$$f_x[t] = 4x_0(t) = 4(t+1), \quad f_y[t] = 2x_0(t-1), \quad f_u[t] = -2u_0(t).$$

Thus, (8) and (9), respectively, have the forms

$$\dot{\delta x}(t) = 4(t+1)\delta x(t) + 2x_0(t-1)\delta x(t-1) - 2\rho_1 x_0(t-1)\dot{x}_0(t-1) - 2\rho_3 \sin(t) u_0(t)$$

and

$$\delta x(t) = 2\rho_2 \cos(t), \quad t \in [-1.5, 0].$$

By elementary calculations we obtain

$$\delta x(t;\delta\mu) = \begin{cases} \delta x_1(t), & t \in [0,1), \\ \delta x_2(t), & t \in [1,2), \end{cases}$$

where

$$\delta x_1(t) = 2\Big\{ e^{2t(t+2)} \Big[\rho_2 + \int\limits_0^t e^{-2s(s+2)} \Big(2\rho_2 \cos(s-1) - \rho_3 \sin(s) \sqrt{2(s+1)^2+1} \Big)\, ds \Big] \Big\},$$

$$\delta x_2(t) = e^{2(t^2+2t-3)} \times \Big\{ \delta x_1(1) + \int\limits_1^t e^{-2(s^2+2s-3)} \Big(2s\delta x_1(s-1) - 2\rho_1 s - 2\rho_3 \sin(s) \sqrt{2(s+1)^2+s^2} \Big)\, ds \Big\}.$$

Consequently, the approximate solution $x(t;\mu)$ of the perturbed equation has the form (see (10))

$$x(t;\mu) \approx t + 1 + \delta x(t;\delta\mu), \quad t \in (0,2).$$

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References

[1] T. Tadumadze, Variation formulas of solutions for functional differential equations with several constant delays and their applications in optimal control problems. *Mem. Differ. Equ. Math. Phys.* **70** (2017), 7–97.