Asymptotic Behavior of Solutions of Third Order Ordinary Differential Equations

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We consider the differential equation

$$y''' = \alpha_0 p(t) y |\ln|y||^\sigma, \tag{1}$$

where $\alpha_0 \in \{-1;1\}$, $p : [a,\omega) \to (0,+\infty)$ is a continuous function, $\sigma \in \mathbb{R}$, $\infty < a < \omega \le +\infty$. It belongs to the equations class of the form

$$y''' = \alpha_0 p(t) L(y), \tag{2}$$

where $\alpha_0 \in \{-1;1\}$,$p : [a,\omega) \to (0,+\infty)$ is a continuous function, $\infty < a < \omega \le +\infty$, the function L is continuous and positive in a one-sided neighborhood of Δ_{Y_0} at points Y_0 (Y_0 equals $\pm\infty$).

For equations of the form (2) in the work of N. Sharay and V. Evtukhov [4] for the function $L(y)$ with rapidly varying nonlinearity it was investigated the question of the existence and asymptotic behavior as $t \to \omega$ of the so-called $P_\omega(Y_0,\lambda_0)$-solution.

In [5, 10] A. Stekhun and V. Evtukhov obtained the results on the existence and asymptotic behavior as $t \to \omega$ of the endangered and unlimited solutions of the differential equation (2), where $L(y) = yL_1(y)$, $L_1(y)$ is a regularly varying function.

For second order equations of the form (1) in the works of V. Evtukhov and M. Jaber [1,2] it was investigated the question on the existence and asymptotic behavior as $t \uparrow \omega$ of all $P_\omega(\lambda_0)$-solutions. It seems natural to try to extend these results to the third-order differential equations.

A solution y of equation (1), specified on the interval $[t_y, w) \subset [a,\omega)$, is said to be a $P_\omega(\lambda_0)$-solution if it satisfies the following conditions:

$$\lim_{t\uparrow\omega} y^{(k)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty \end{cases} (k = 0,1,2), \quad \lim_{t\uparrow\omega} \frac{[y''(t)]^2}{y'''(t)y'(t)} = \lambda_0.$$

In the work [3] it is shown that a set of $P_\omega(\lambda_0)$-solutions with regards to their asymptotic properties in the five class solutions, corresponding values $\lambda_0 \in \mathbb{R}\backslash\{0;1;\frac{1}{2}\}$, $\lambda_0 = \pm\infty$, $\lambda_0 = 0$, $\lambda_0 = \frac{1}{2}$ and $\lambda_0 = 1$.

Earlier in [7–9] the results were obtained in the case, when $\lambda_0 \in R\backslash\{0,\pm1,\frac{1}{2}\}$ and $\lambda_0 = \pm\infty$. The goal of the work is the establishment existence conditions for equation (1) of $P_\omega(1)$-solutions and also asymptotic representations as $t \uparrow \omega$ of such solutions and their derivatives of second order.

We introduce the necessary notation.

$$\pi_\omega(t)=\begin{cases} t & \text{if } \omega=+\infty,\\ t-\omega & \text{if } \omega<+\infty, \end{cases}\quad I_B(t)=\int_B^t p^{\frac13}(\tau)\,d\tau,\quad B=\begin{cases} a & \text{if } \int\limits_a^\omega p^{\frac13}(\tau)\,d\tau=+\infty,\\ \omega & \text{if } \int\limits_a^\omega p^{\frac13}(\tau)\,d\tau<+\infty. \end{cases}$$

Theorem 1. *Let $\sigma\neq 3$, the function $p:[a,\omega)\to(0,+\infty)$ be continuously differentiable and there exist a finite or equal to $\pm\infty$ limit*

$$\lim_{t\uparrow\omega}\frac{(p^{\frac13}(t)|I_B(t)|^{\frac{\sigma}{2-\sigma}})'}{p^{\frac23}(t)|I_B(t)|^{\frac{2\sigma}{3-\sigma}}}. \tag{3}$$

For the existence of $P_\omega(1)$-solutions of equation (1) it is necessary and sufficient the conditions

$$\alpha_0>0 \ \ and \ \ \lim_{t\uparrow\omega}\pi_\omega(t)p^{\frac13}(t)|I_B(t)|^{\frac{\sigma}{3-\sigma}}=\infty \tag{4}$$

to hold. Moreover, for each such solution there take place the following asymptotic representations as $t\uparrow\omega$

$$\ln|y(t)|=\mu\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{3}{3-\sigma}}[1+o(1)],\quad \frac{y'(t)}{y(t)}=p^{\frac13}(t)\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{\sigma}{3-\sigma}}[1+o(1)],$$
$$\frac{y''(t)}{y(t)}=p^{\frac13}(t)\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{\sigma}{3-\sigma}}[1+o(1)],$$

where $\mu=\operatorname{sign}(\frac{3-\sigma}{3}I_B(t))$.

Theorem 2. *Let $\sigma\neq 3$, the function $p:[a,\omega)\to(0,+\infty)$ be continuously differentiable and along with (3), (4) the following condition*

$$\lim_{t\uparrow\omega}\frac{(p^{\frac23}(t)|I_B(t)|^{\frac{2\sigma}{3-\sigma}})'}{p(t)|I_B(t)|^{\frac{3(\sigma-1)}{3-\sigma}}}=0$$

hold. Then for any $C=\pm1$ equation (1) has a $P_\omega(1)$-solution. Furthermore, for every such solution the following asymptotic representations as $t\to\omega$

$$y(t)=C\exp\Big[\mu\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{3}{3-\sigma}}\Big]\,[1+o(1)],\quad y'(t)=\mu p^{\frac13}(t)\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{\sigma}{3-\sigma}}y(t)[1+o(1)],$$
$$y''(t)=\mu p^{\frac23}(t)\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|^{\frac{2\sigma}{3-\sigma}}y(t)[1+o(1)]$$

take place.

We give a corollary of these theorems, when $\sigma=0$, i.e. for the following linear differential equation

$$y'''=\alpha_0p(t)y, \tag{5}$$

where $\alpha_0\in\{-1;1\}$, $\sigma\in\mathbb{R}$, $p:[a,w)\to(0,+\infty)$ is a continuous function, $a<w\le+\infty$.

Corollary 1. *Let the function $p:[a,\omega)\to(0,+\infty)$ be continuously differentiable and there exist a finite or equal to $\pm\infty$ limit $\lim\limits_{t\uparrow\omega} p'(t)p^{-\frac{5}{3}}(t)$. For the existence of $P_\omega(1)$-solutions of equation* (5) *it is necessary and sufficient the conditions*

$$\alpha_0>0 \ \text{ and } \ \lim_{t\uparrow\omega}\pi_\omega^3(t)p(t)=+\infty \tag{6}$$

to hold. Moreover, for each such solution the following asymptotic representations as $t\uparrow\omega$

$$\ln|y(t)|=\mu\Big|\frac{3-\sigma}{3}\,I_B(t)\Big|[1+o(1)],\quad \frac{y'(t)}{y(t)}=p^{\frac13}(t)[1+o(1)],\quad \frac{y''(t)}{y(t)}=p^{\frac13}(t)[1+o(1)],$$

where $\mu=\operatorname{sign}(I_B(t))$, take place.

Corollary 2. *Let the function $p:[a,\omega)\to(0,+\infty)$ be continuously differentiable and along with conditions* (6) *the following condition is satisfied*

$$\int_a^\omega\Big|\frac{p'(t)}{p(t)}\Big|\,dt<+\infty.$$

Then equation (5) *has a $P_\omega(1)$-solution. Furthermore, for any such solution the following asymptotic representations as $t\to\omega$*

$$y_i(t)=\exp\big[(-1)^{i-1}I_B(t)\big][1+o(1)],\quad y'(t)=(-1)^{i-1}p^{\frac13}(t)y(t)[1+o(1)],$$
$$y''(t)=(-1)^{i-1}p^{\frac23}(t)y(t)[1+o(1)]\ \ (i=1,2,3)$$

take place.

The obtained results are consistent with the already known results for linear differential equations (see [6, Chapter 1]).

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