On Existence of Solutions with Prescribed Number of Zeros to Emden–Fowler Equations with Variable Potential

V. V. Rogachev

Lomonosov Moscow State University, Moscow, Russia
E-mail: valdakhar@gmail.com

1 Introduction

The problem of the existence of solutions to Emden–Fowler type equations with prescribed number of zeros on a given domain is studied.

Consider the equation

$$y^{(n)}+p(t,y,y',\dots,y^{n-1})|y|^k\operatorname{sgn}y=0,\ \ k\in(0,1)\cup(1,\infty). \tag{1.1}$$

We say that $p\in\mathfrak{P}_n$ if for some $m,M\in\mathbb{R}$ the inequalities $0<m\le p(t,\xi_1,\xi_2,\dots,\xi_n)\le M<\infty$ hold, the function $p(t,\xi_1,\xi_2,\dots,\xi_n)$ is continuous and Lipschitz continuous in $(\xi_1,\xi_2,\dots,\xi_n)$.

We prove that this equation with $p\in\mathfrak{P}_n$ has a solution with a given finite number of zeros on a given interval. Results considering the existence of solutions with countable number of zeros are presented in [9,10]. For the equation (1.1) with $n=3,4$ and constant potential $p=p_0$ the existence of solutions with a given finite number of zeros on a given interval is proved in [4], and for the case $n=3$, $p\in\mathfrak{P}_n$ – in [5,7]. Now we generalise this result for $n>3$, $p\in\mathfrak{P}_n$.

2 Main result

Theorem 2.1. *For any $k\in(0,1)\cup(1,\infty)$, $n\ge3$, $p\in\mathfrak{P}_n$, $[a,b]\subset\mathbb{R}$, and integer $S\ge2$, equation* (1.1) *has a solution defined on the segment $[a,b]$, vanishing on its end points a,b, and having exactly S zeros on $[a,b]$.*

3 Sketch of the proof

3.1 The case of constant potential

In the case of constant potential p proof is based on the following theorems.

Theorem 3.1 ([3], [1, Theorem 5])**.** *For any $n>2$ and real $k>1$ there exists a non-constant oscillatory periodic function h such that for any $p_0\in\mathbb{R}$ with $p_0>0$ and any $t^*\in\mathbb{R}$ the function*

$$y(t)=|p_0|^{\frac{1}{1-k}}(t^*-t)^{-\alpha}h(\log(t^*-t)),\ \ -\infty<t<t^*,\ \ \alpha=\frac{n}{k-1}\,,$$

is a solution to equation (1.1) *with constant potential $p=p_0$.*

Theorem 3.2 ([1, Theorem 9])**.** *For any $n>2$ and real $k\in(0,1)$ there exists a non-constant oscillatory periodic function h such that for any $p_0\in\mathbb{R}$ with $(-1)^np_0>0$ and any $t^*\in\mathbb{R}$ function*

$$y(t)=|p_0|^{\frac{1}{1-k}}(t^*-t)^{-\alpha}h(\log(t^*-t)),\ \ -\infty<t<t^*,\ \ \alpha=\frac{n}{k-1},$$

is a solution to equation (1.1) *with constant potential $p=p_0$.*

Lemma 3.1 ([2, Lemma 6.1])**.** *If $y(t)$ is a solution to equation* (1.1) *with constant potential $p = p_0$, and constants A, B, C satisfy $|A| = B^{\frac{n}{k-1}}$, $B > 0$, then $z(t) = Ay(Bt + C)$ is also a solution to the same equation.*

From theorems 3.1 and 3.2 it follows that equation (1.1) with constant potential $p = p_0$ has a solution $y(t)$ with countable number of zeros. Then it is possible to choose segment $[t_1, t_2]$ where $y(t_1) = y(t_2) = 0$ and $y(t)$ has exactly S zeros on the segment. Then, due to lemma 3.1, function

$$\widetilde{y}(t) = \Big(\frac{|t_2 - t_1|}{|b - a|}\Big)^{\frac{n}{k-1}} y\Big(x_1 + \frac{|t_2 - t_1|}{|b - a|}\,(t - a)\Big), \tag{3.1}$$

is a solution to the equation, it is defined on the segment $[a, b]$, $y(a) = 0$, $y(b) = 0$, and $y(t)$ has exactly S zeros on $[a, b]$. When n is odd, we use substitution $t \mapsto -t$ to consider p_0 with opposite sign. This completes the proof in the case of constant potential.

3.2 The case of variable potential

It is impossible to use same methods to prove main theorem when $p \in \mathfrak{P}_n$. The full proof of the main theorem is given in [8] (the case $k \in (1, \infty)$) and in [6] (the case $k \in (0, 1)$). The proof is based on the following results.

Lemma 3.2 (generalisation of [2, Lemma 7.1])**.** *If $y(t)$ is a solution to* (1.1) *satisfying, at some t_0, the conditions*

$$y(t_0) \ge 0, y'(t_0) > 0, y''(t_0) \ge 0, \dots, y^{(n-1)}(t_0) \ge 0,$$

then at some $t_0' > t_0$ the solution has a local maximum and satisfies

$$\begin{aligned} t_0' - t_0 &\le (\mu y'(t_0))^{-\frac{k-1}{k+n-1}}, \\ y(t_0') &> (\mu y'(t_0))^{\frac{n}{k+n-1}}, \end{aligned}$$

where the constant $\mu > 0$ depends only on n, k, m, M.

Lemma 3.3 (generalisation of [2, Lemma 7.2])**.** *If $y(t)$ is a solution to* (1.1) *satisfying, at some t_0', the conditions*

$$y(t_0') > 0, y'(t_0') \le 0, \dots, y^{(n-1)}(t_0') \le 0,$$

then at some $t_0 > t_0'$ the solution is equal to zero, and

$$\begin{aligned} t_0 - t_0' &\le (\mu y(t_0'))^{-\frac{k-1}{n}}, \\ y'(t_0) &< -(\mu y(t_0'))^{\frac{k+n-1}{n}}, \end{aligned}$$

where the constant $\mu > 0$ depends only on n, k, m, M.

Lemma 3.4 (generalisation of [2, Lemma 7.3])**.** *Under the assumptions of Lemmas 3.2 and 3.3, for any $t_1 > t_0$ with $y(t_0) = 0$, $y(t_1) = 0$ the inequality*

$$|y'(t_1)| > Q|y'(t_0)|$$

holds true, where the constant $Q > 1$ depends only on k, m, M.

Lemma 3.5 ([5, 8])**.** *Suppose $D \subset \mathbb{R}^n$ and $\widetilde{D} \subset \mathbb{R}^{n+1}$ are open connected sets such that for every $c \in D$ there exists a segment $[0, x_c]$ with $[0, x_c] \times \{c\} \subset \widetilde{D}$. Suppose that $f(x, c)$ is a continuous function $\widetilde{D} \to \mathbb{R}$ as well as its derivative in x. Suppose that for every $c \in D$ the following conditions are fulfilled.*

- *$f(0,c)=0$.*
- *There exists $x_1(c)\in(0,x_c)$ such that $f(x_1(c),c)=0$ and $f(x,c)\neq 0$ for all $x\in(0,x_1(c))$.*
- $$f'_x(x,c)\big|_{x=0}\neq 0,\quad f'_x(x,c)\big|_{x=x_1(c)}\neq 0.$$

Then $x_1(c)$ is a continuous function $D\to\mathbb{R}$.

In the case $k>1$ the main result is proved as follows. We consider a solution $y(t)$ with initial values
$$y(a)=0, y'(a)=y_1, y''(a)=y_2,\dots,y^{(n-1)}(a)=y_{n-1},$$
where $y_i>0$, $i=1,\dots,n-1$. Due to Lemmas 3.2–3.4, the solution $y(t)$ oscillates; so, $y(t)$ has a sequence of zeros t_j, $j\in\mathbb{N}$. We consider the position of a particular zero t_{S-1} as a function of initial values $y_1,\dots,y_{n-1}$, and with the help of Lemma 3.5 we find out that this function is continuous. Then, obtaining some estimates, we prove that the range of values of $t_{S-1}(y_1,\dots,y_{n-1})$ is $(a,+\infty)$, and that means that for some initial values we have $t_{S-1}=b$, whence the corresponding solution $y(t)$ has exactly S zeros on $[a,b]$.

In the case $k\in(0,1)$ the same methods apply, but equation (1.1) with $k\in(0,1)$ does not satisfy the conditions of the theorem of continuous dependence of solutions to ODE, which was used in the proof. We have to find a workaround here, and it is provided by the following lemmas (see [6]), which act as replacements for the mentioned continuous dependence theorem.

Lemma 3.6 ([6])**.** *Suppose that $n\geq 3$, $k\in(0,1)$, $p\in\mathfrak{P}_n$, and y is a solution to*
$$y^{(n)}+p(t,y,y',\dots,y^{(n-1)})|y|^k\operatorname{sgn}y=0,\ \ y^{(i)}(t_0)=y_i,\ \ i=\overline{0,n-1},$$
defined on $[a,b]$. In addition, suppose that for some $w\in\mathbb{R}$ the inequality $|y'|\geq w>0$ holds true on $[a,b]$. Then there exists $v\in\mathbb{R}^+$ such that for every $I=[t_0,t^*]\subset[a,b]$ with $|I|<v$, for every $\varepsilon>0$ there exists $\delta>0$ such that if some $q\in\mathfrak{P}_n$, $z_i\in\mathbb{R}$, $i=\overline{0,n-1}$, satisfy the inequalities
$$|p-q|<\delta,\ \ |z_i-y_i|<\delta,\ \ i=\overline{0,n-1},$$
and z is a solution to
$$z^{(n)}+q(t,z,z',\dots,z^{(n-1)})|z|^k\operatorname{sgn}z=0,\ \ z^{(i)}(t_0)=z_i,\ \ i=\overline{0,n-1},$$
then z is defined on or can be extended onto I with the inequalities
$$\big|z^{(i)}(t)-y^{(i)}(t)\big|<\varepsilon,\ \ i=\overline{0,n-1},$$
satisfied on it.

Lemma 3.7 ([6])**.** *Suppose that $n\geq 3$, $k\in(0,1)$, $p\in\mathfrak{P}_n$, and y is a solution to*
$$y^{(n)}+p(t,y,y',\dots,y^{(n-1)})|y|^k\operatorname{sgn}y=0,\ \ y^{(i)}(t_0)=y_i,\ \ i=\overline{0,n-1},$$
defined on $[a,c]$, and y has a finite number of zeros, all of them being of first order. Then for every $\varepsilon>0$ there exists $\delta>0$ such that if some $q\in\mathfrak{P}_n$, $z_i\in\mathbb{R}$, $i=\overline{0,n-1}$, satisfy the inequalities
$$|p-q|<\delta,\ \ |z_i-y_i|<\delta,\ \ i=\overline{0,n-1},$$
and z is a solution to
$$z^{(n)}+q(t,z,z',\dots,z^{(n-1)})|z|^k\operatorname{sgn}z=0,\ \ z^{(i)}(t_0)=z_i,\ \ i=\overline{0,n-1},$$
then z is defined on or can be extended onto $[a,c]$ with the inequalities
$$\big|z^{(i)}(t)-y^{(i)}(t)\big|<\varepsilon,\ \ i=\overline{0,n-1},$$
satisfied on it.

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